Open problems session

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May 23, 2022

Partitions mod 2 and 3

Theorem (Ramanujan's Congruences 1919; Hardy-Ramanujan)

$$p(5n+4) \equiv 0 \pmod 5, \qquad p(7n+5) \equiv 0 \pmod 7,$$

$$p(11n+6) \equiv 0 \pmod{11}.$$

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Theorem (Radu (2012))

No linear congruences exist for partitions modulo 2 or 3.

Partitions mod 2 and 3 (continued)

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Every arithmetic progression contains infinitely many odd and infinitely many even partition values.

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Are these infinitely sets of even or odd values actually density 1/2? Can one even show that the density of all even or odd partition numbers is even positive? (Fundamental barrier: $X^{\frac{1}{2}+\varepsilon}$ odd/even values up to X). What can one say about partitions mod 3?

A combinatorial realization?

Definition (Dyson 1944)

$\mathrm{rank}(\lambda) =$ *largest part* λ_1 $-$ $\#$ *of parts* k.

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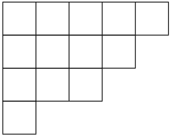
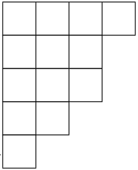
λ=5+4+3+1
(rank =1)



$\to$ λ'=4+3+3+2+1
(rank =−1)



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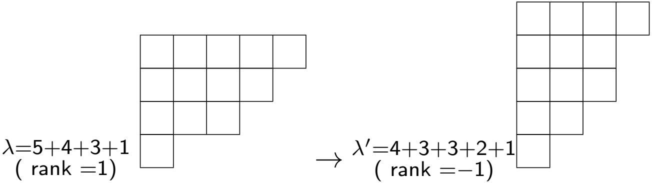

- $N(m,n) := \#\{\text{ptns of } n \text{ with rank } m\}$,
  $N(m,q;n) := \#\{\text{ptns of } n \text{ with rank } \equiv m \pmod{q}\}$.

Dyson's Conjecture

Theorem (Conjecture of Dyson 1944, proven by Atkin and Swinnerton-Dyer in 1954)

We have

$$N(0,5;5n+4) = N(1,5;5n+4) = \ldots = N(4,5;5n+4).$$

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Similarly for ranks mod 7 for partitions of $7n+5$.

- This "explains" Ramanujan's congruences mod 5 and 7 using a combinatorial object.

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Definition (Andrews-Garvan, 1988)

$$\mathit{crank}(\lambda) := \begin{cases} \textit{largest part of } \lambda & \textit{if no 1's in } \lambda, \\ (\#\ \textit{parts larger than}\ \#\ \textit{of 1's}) - (\#\ \textit{of 1's}) & \textit{else.} \end{cases}$$

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Theorem (Andrews-Garvan)

Cranks "explain" Ramanujan's congruences mod 5, 7, *and* 11.

Reframing the combinatorial proofs

Elementary Fact

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Elementary Fact

The equidistribution for cranks mod ℓ on a progression $\ell n + \beta$ is equivalent to

$$\Phi_\ell(\zeta) | [q^{\ell n+\beta}] C(z; \tau).$$

Here, Φ_ℓ is the ℓ-th cyclotomic polynomial, and divisibility is as Laurent polynomials.

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Question (Stanton)

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- Stanton first notes the divisibility $\Phi_\ell(\zeta) | [q^{\ell n+\beta}]R/C(z;\tau)$.
- If the quotient had *positive* coefficients, he suggested they may count something important.
- This doesn't work directly.

Stanton's Conjecture

Definition (Stanton)

The **modified rank and crank** *are:*

$$rank^*_{\ell,n}(\zeta) := rank_{\ell n+\beta} + \zeta^{\ell n+\beta-2} - \zeta^{\ell n+\beta-1} + \zeta^{2-\ell n-\beta} - \zeta^{1-\ell n-\beta},$$

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where $\beta := \ell - \frac{\ell^2-1}{24}$.

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Conjecture (Stanton)

All of the following are Laurent polynomials with non-negative coefficients:

$$\frac{rank^*_{5,n}(\zeta)}{\Phi_5(\zeta)}, \quad \frac{rank^*_{7,n}(\zeta)}{\Phi_7(\zeta)}, \quad \frac{crank^*_{5,n}(\zeta)}{\Phi_5(\zeta)}, \quad \frac{crank^*_{7,n}(\zeta)}{\Phi_7(\zeta)}, \quad \frac{crank^*_{11,n}(\zeta)}{\Phi_{11}(\zeta)}.$$

Result for cranks

Theorem (Bringmann, Gomez, Rolen, Tripp, 2021)

The crank part of Stanton's Conjecture is true.

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Result for cranks

Theorem (Bringmann, Gomez, Rolen, Tripp, 2021)

The crank part of Stanton's Conjecture is true.

Question

What about for ranks? What do the positive numbers mean for cranks? How does this generalize?

Modulo 9 Kanade-Russell conjectures

Conjecture (Kanade-Russell, 2015)

One of the Kanade-Russell conjectures is

$$\#\{\lambda \vdash n : \lambda_i \equiv \pm 1, \pm 3 \pmod 9\}$$
$$= \#\left\{\lambda \vdash n : \substack{\lambda_i - \lambda_{i+1} \le 1 \Rightarrow \lambda_i + \lambda_{i+1} \equiv 0 \pmod 3 \\ \lambda_i - \lambda_{i+2} \ge 3}\right\}.$$

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The associated q-series identity is

$$\sum_{m,n\geq 0} \frac{q^{m^2+3mn+3n^2}}{(q;q)_m (q^3;q^3)_n} = \frac{1}{(q,q^3,q^6,q^8;q^9)_\infty}.$$

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Remark

- *Sum-product identities*

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Remark

- *Sum-product identities*
- *Connection to level 2 affine Lie algebra characters*

Open q-series questions of Andrews

Definition

Let

$$v_1(q) := 1 + \sum_{n\geq 1} \frac{q^{\frac{n(n+1)}{2}}}{(-q^2;q^2)_n} = \sum_{n\geq 0} V_1(n)q^n.$$

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- Let $OE(n)$ be the number of partitions of n in which the parity of the parts alternates and the smallest part is odd.

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Combinatorial interpretation

- Let $OE(n)$ be the number of partitions of n in which the parity of the parts alternates and the smallest part is odd.
- The rank of such a partition is even.
- $V_1(n)$ is the number odd-even partitions of n with rank $\equiv 0 \pmod 4$ minus the number with rank $\equiv 2 \pmod 4$.

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Conjecture (Andrews, 1986)

1. $|V_1(n)| \to \infty$ *as* $n \to \infty$.

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3. *For* $n \geq 5$ *there is an infinite sequence* $N_5 = 293, N_6 = 410, \ldots, N_{28} = 7898, \ldots$ *such that* $V_1(N_n), V_1(N_n+1)$, *and* $V_1(N_n+2)$ *all have the same sign.*

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Remark

Andrews also gives functions $v_2(q), v_3(q), v_4(q)$ *for which similar conjectures exist.*

Strongly unimodal sequences with fixed rank

Definition

A sequence of positive integers $\{a_j\}_{j=1}^{s}$ is **strongly unimodal of size** n if it satisfies

1. $a_1 < \cdots < a_{k-1} < a_k > a_{k+1} > \cdots > a_s$,
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Definition

- The rank of a strongly unimodal sequence is the number of terms after the maximal term minus the number of terms that precede it, i.e. the rank is $s - 2k + 1$.
- Let $u(m, n)$ be the number of strongly unimodal sequences of size n and rank m.

Strongly unimodal sequences with fixed rank

Theorem (Bringmann–Jennings-Shaffer-Mahlburg-Rhoades, 2018)

For a fixed $m \in \mathbb{N}_0$,

$$U_m(q) = \sum_{n \geq 1} u(m,n) q^n$$
$$= \frac{q^{\frac{m(m+1)}{2}}}{(q)_\infty} \sum_{n \geq 1} \frac{(-1)^n q^{\frac{n(n+1)}{2} + mn}}{1 - q^{n+m}} \left(q^{n(n+m)} - 1\right).$$

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Corollary

We have the indefinite theta representation

$$V_m(q) = (q)_\infty U_m(q)$$
$$= \sum_{n_1,n_2\geq 0} (-1)^{n_1+n_2} q^{\frac{1}{2}\left(n_1+m+\frac{1}{2}\right)^2+\frac{3}{2}\left(n_2+\frac{1}{2}\right)^2+2\left(n_1+m+\frac{1}{2}\right)\left(n_2+\frac{1}{2}\right)}.$$

Strongly unimodal sequences with fixed rank

Open problem

Determine the (generalized) quantum modular properties of $U_m(q)$ or $V_m(q)$.

Mock Maass theta functions

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- Example from Ramanujan's Lost Notebook studied by Andrews-Dyson-Hickerson and Cohen:

$$\sigma(q) := \left(\sum_{\substack{n+j\geq 0 \\ n-j\geq 0}} + \sum_{\substack{n+j<0 \\ n-j<0}} \right) (-1)^{n+j} q^{\frac{3}{2}(n+1/6)^2-j^2}.$$

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- Note: If you change signs to replace $\left(\sum_{\substack{n+j\geq 0 \\ n-j\geq 0}} + \sum_{\substack{n+j<0 \\ n-j<0}} \right)$ with $\left(\sum_{\substack{n+j\geq 0 \\ n-j\geq 0}} - \sum_{\substack{n+j<0 \\ n-j<0}} \right)$, then get (essentially) a sixth order mock theta function of Ramanujan.

Mock Maass theta functions (cont.)

- Zwegers gave a general construction of “mock Maass theta functions” Φ, for these kind of indefinite theta series; when q^n is replaced by $e^{2\pi iux}K_0(2\pi ivn)$ (this makes it have eigenvalue $1/4$ under Δ_0), then it is “almost” modular.

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- Namely, one can add a special integral to it to "complete" the function to the modular function $\hat{\Phi}$. But instead of being an eigenfunction of Δ_0, applying $\Delta_0 - 1/4$ gives you stuff like cusp forms times complex conjugates of cusp forms.

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- Namely, one can add a special integral to it to "complete" the function to the modular function $\hat{\Phi}$. But instead of being an eigenfunction of Δ_0, applying $\Delta_0 - 1/4$ gives you stuff like cusp forms times complex conjugates of cusp forms.
- In analogy to harmonic Maass forms, "holomorphic" is replaced by "eigenvalue $1/4$", and the period integrals are replaced with new similar integrals.

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Question

What else can be done with this theory?

Sample place to look

- Sample place to look: 4 families of Maass form "q-functions" from this theory in Bringmann-Lovejoy-Rolen, including:

$$\sum_{n\geq 0}(q)_n(-1)^n q^{\binom{n+1}{2}} H_n(k,\ell;0,q),$$

where

$$H_n(k,\ell;b;q) := \sum_{n=n_k\geq n_{k-1}\geq \ldots \geq n_1 \geq 0} \prod_{j=1}^{k-1} q^{n_j^2+(1-b)n_j}$$

$$\times \begin{bmatrix} n_{j+1}-n_j-bj+\sum_{r=1}^{j}(2n_r+\chi_{\ell>r}) \\ n_{j+1}-n_j \end{bmatrix}_q .$$

Explicit questions about these families of q-series

Question

These are positive coefficients of Maass forms; like $\sigma(q)$.

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These are positive coefficients of Maass forms; like $\sigma(q)$. Are there q-hypergeometric formulas for the negative coefficients?

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These are positive coefficients of Maass forms; like $\sigma(q)$. Are there q-hypergeometric formulas for the negative coefficients? Related examples of Li-Ngo-Rhoades had such a feature, and this could shed light on quantum modular properties as $q \mapsto q^{-1}$ that they noted; are there Bailey pairs that work?

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Returning to the $U_m(q)$ functions

- When applying Zwegers machinery to $V_m(q)$, Φ is not modular so we complete it to $\hat{\Phi}$.

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General open problem

In the case of Zwegers' construction when $\Phi \neq \hat{\Phi}$, determine the generalized quantum modular properties of Φ^+ and Φ^-.

Higher depth mock modular forms

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Are there nice combinatorial/q-hypergeomtric examples of higher depth forms?

Sums of roots of unity

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By studying radial limits of mock theta funtions as Ramanujan did in his final letter to Hardy, one can find strange identities of sums of roots of unity, like:

$$\sum_{n=0}^{\frac{k-2}{4}} \frac{\zeta_k^{hn}(-\zeta_k^h;\zeta_k^{2h})_n}{(\zeta_k^h;\zeta_k^{2h})_{n+1}} = i\sum_{n=0}^{\frac{k}{2}-1} \frac{(-1)^{\frac{n(n+1)}{2}}\zeta_k^{hn(n+1)}(\zeta_k^{2h};-\zeta_k^{2h})_n}{(i\zeta_k^h;-\zeta_k^{2h})_{n+1}^2}.$$

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Can one prove this directly?

Congruences modulo powers of 2

- Ramaujan's mock theta function:

$$\omega(q) = \sum_{n\geq 0} a_\omega(n) q^n := \sum_{n\geq 0} \frac{q^{2n(n+1)}}{(q;q^2)^2_{n+1}}.$$

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- From Borcherds' products, Bruinier-Ono define the "sieved log-derivative" $\widetilde{L}_\omega(q) = \sum_{\substack{n\geq 1 \\ (n,6)=1}} \widehat{\sigma}_\omega(n) q^n$, where

$$\widehat{\sigma}_\omega(n) := \sum_{d|n} \left(\frac{d}{3}\right)\left(\frac{-8}{n/d}\right) d \cdot a_\omega\left(\frac{2d^2-2}{3}\right).$$

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Theorem (Bruinier-Ono, 2010)

We have that $\widetilde{L}_\omega(q) \equiv \sum_{(n,6)=1} \sigma_1(n)q^n \pmod{512}$.

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- Strongly unimodal sequences rank generating function:

$$\mathcal{U}(\zeta;q) := \sum_{m,n} u(m,n)\zeta^m q^n = \sum_{n\geq 0} (-\zeta q)_n (-\zeta^{-1})_n q^{n+1}.$$

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Theorem (Bryson-Pitman-Ono-Rhoades, Chen-Garvan)

If $\ell \equiv 7, 11, 13, 17 \pmod{24}$ is prime with $\left(\frac{k}{\ell}\right) = -1$, then for all n, $u(\ell^2 n + k\ell - (\ell^2-1)/24) \equiv 0 \pmod{4}$.

Unimodal sequences (cont.)

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