Characters, Schemes and q-series

Antun Milas, SUNY-Albany

100 Years of Mock Theta Functions (Vanderbilt)
May 2022

This talk

This talk

Part I: q-series (identities) from graphs and commutative algebras.

This talk

Part I: q-series (identities) from graphs and commutative algebras.

Part II: q-series from Schur's indices of 4d $\mathcal{N} = 2$ SCFTs.

This talk

Part I: q-series (identities) from graphs and commutative algebras.

Part II: q-series from Schur's indices of 4d $\mathcal{N} = 2$ SCFTs.

Part III: (Time permitting) Generalized multiple q-zeta values

This talk

Part I: q-series (identities) from graphs and commutative algebras.

Part II: q-series from Schur's indices of 4d $\mathcal{N} = 2$ SCFTs.

Part III: (Time permitting) Generalized multiple q-zeta values

References

Main references:

A.M. arXiv 2203.15642

and joint papers:

Jennings-Shaffer- A.M. 2019,2020

Bringmann-Jennings-Shaffer-A.M. 2021

Li-A.M. 2020

Kanade A. M. Russell, 2021

Part I

Graph Series

Graph Series

Definition (Graph series)

Given an undirected simple graph Γ with r nodes. Let $E(\Gamma)$ denotes the set of edges of Γ. The q-series

$$H_\Gamma(q) = \sum_{n_1,\dots,n_r \geq 0} \frac{q^{n_1+\cdots+n_r+\frac{1}{2}\mathrm{n}C\mathrm{n}^T}}{(q)_{n_1}\cdots(q)_{n_r}},$$

where C is the adjacency matrix of Γ, is called graph q-series of Γ. If $(i,j) \in E(\Gamma)$ then $\frac{1}{2}\mathrm{n}C\mathrm{n}^T$ contributes with $n_i n_j$ in the exponent.

Examples

(i) • (single node and no edges):

$$H_\Gamma(q) = \sum_{n\geq 0} \frac{q^n}{(q)_n} \stackrel{Euler}{=} \frac{1}{(q)_\infty}.$$

(ii) • − •

$$H_\Gamma(q) = \sum_{n_1,n_2\geq 0} \frac{q^{n_1+n_2+n_1 n_2}}{(q)_{n_1}(q)_{n_2}}$$

(iii) 3-cycle

$$H_\Gamma(q) = \sum_{n_1,n_2,n_3\geq 0} \frac{q^{n_1+n_2+n_3+n_1 n_2+n_2 n_3+n_3 n_1}}{(q)_{n_1}(q)_{n_2}(q)_{n_3}}$$

Convergence

Convergence

Observe that for many graphs (e.g. simple graphs)

$$\sum_{n_1,\ldots,n_r\geq 0} \frac{q^{\frac{1}{2}\mathrm{n}C\mathrm{n}^T}}{(q)_{n_1}\cdots(q)_{n_r}},$$

doesn't converge inside $|q|<1$. So it is important to shift

$$H_\Gamma(q) = \sum_{n_1,\ldots,n_r\geq 0} \frac{q^{\frac{1}{2}\mathrm{n}C\mathrm{n}^T+n_1+\cdots+n_r}}{(q)_{n_1}\cdots(q)_{n_r}},$$

now convergent for all Γ. Instead, we can consider

$$H_\Gamma(q,x) = \sum_{n_1,\ldots,n_r\geq 0} \frac{q^{\frac{1}{2}\mathrm{n}C\mathrm{n}^T}x^{n}}{(q)_{n_1}\cdots(q)_{n_r}}.$$

Graphs series vs. Nahm's sums

Graphs series vs. Nahm's sums

Given **positive definite** $r \times r$ integral matrix A, and $B \in \mathbb{Z}^r$ (Nahm sum):

$$f_{A,B}(q) = \sum_{n_1,\ldots,n_r \geq 0} \frac{q^{\frac{1}{2}nAn^T + B\cdot n}}{(q)_{n_1} \cdots (q)_{n_r}}$$

These series are often associated to ADE type Dynkin diagrams $\rightsquigarrow$ famous ADE q-series identities entering various combinatorial identities (e.g. Rogers-Ramanujan identities). But the quadratic form does not come from the incidence matrix but instead from (Euler/Tits quadratic form):

$$A := 2I_r - C$$

Example. Nahm sum associated to A_2 Dynkin diagram $\bullet - \bullet$ is

$$\sum_{n_1,n_2 \geq 0} \frac{q^{n_1^2+n_2^2-n_1 n_2}}{(q)_{n_1}(q)_{n_2}}$$

Graph series from geometry

Graph series from geometry

Consider

$$R = \frac{\mathbb{C}[x_1, x_2, \dots, x_n]}{(f_1, f_2, \dots, f_k)}$$

where f_i are homogeneous.Then R is also graded, $R = \oplus_{n \geq 0} R(n)$. We can define its **Hilbert series** $H_R(t) = \sum_{n \geq 0} dim(R(n))t^n$.

Graph series from geometry

Consider

$$R = \frac{\mathbb{C}[x_1, x_2, \dots, x_n]}{(f_1, f_2, \dots, f_k)}$$

where f_i are homogeneous.Then R is also graded, $R = \oplus_{n\geq 0} R(n)$. We can define its **Hilbert series** $H_R(t) = \sum_{n\geq 0} dim(R(n))t^n$.
With standard grading $deg(x_i) = 1$, we have

$$H_R(t) = \frac{p(t)}{(1-t)^n} = \frac{h(t)}{(1-t)^k}$$

k, dimension of R and $h(t)$, $h(1) \neq 0$ is so called h-polynomial .

Example

$R = k[x, y]/(xy)$.

$$H_R(t) = \frac{1-t^2}{(1-t)^2} = \frac{1+t}{1-t}$$

Graph series from geometry

Consider

$$R = \frac{\mathbb{C}[x_1, x_2, \ldots, x_n]}{(f_1, f_2, \ldots, f_k)}$$

where f_i are homogeneous.Then R is also graded, $R = \oplus_{n \geq 0} R(n)$. We can define its **Hilbert series** $H_R(t) = \sum_{n \geq 0} dim(R(n)) t^n$.
With standard grading $deg(x_i) = 1$, we have

$$H_R(t) = \frac{p(t)}{(1-t)^n} = \frac{h(t)}{(1-t)^k}$$

k, dimension of R and $h(t)$, $h(1) \neq 0$ is so called h-polynomial .

Example

$R = k[x, y]/(xy)$.

$$H_R(t) = \frac{1-t^2}{(1-t)^2} = \frac{1+t}{1-t}$$

$$0 \to k[x, y] \xrightarrow{\cdot xy} k[x, y] \to k[x, y]/(xy) \to 0$$

m-Jet algebras/schemes and arc algebras

m-Jet algebras/schemes and arc algebras

Let f_i be polynomials. Consider

$$R = \frac{\mathbb{C}[x_1, x_2, \ldots, x_n]}{(f_1, f_2, \ldots, f_k)}.$$

$$J_m(R) := \frac{\mathbb{C}[x_{j,(i)} \mid 0 \leq i \leq m,\ 1 \leq j \leq n]}{(D^j f_i | i = 1, \ldots k,\ j \in \mathbb{N})},$$

$$D(x_{j,(i)}) := \begin{cases} x_{j,(i+1)} & \text{for } 0 \leq i \leq m-1 \\ 0 & \text{for } i = m. \end{cases}$$

called the **algebra of m-jets** of R. Let $X_m = \mathrm{Spec}(R_m)$.
$X_\infty = \varprojlim_m X_m$ is called the arc space of $X = \mathrm{Spec}(R)$.
$J_\infty(R) := R_\infty$, the **arc algebra** of R.

Hilbert series

Assuming $(f_1, ..., f_k)$ is homogeneous, letting

$$deg(x_{i,(j)}) = j + 1$$

then $J_m(R)$ and $J_\infty(R)$ are also graded and we can define Hilbert-Poincaré series

$$H_q(J_\infty(R)) = \sum_{j \geq 0} \dim(J_\infty(R))_j q^j$$

Example

$R = k[x_1, ..., x_n]$.

Hilbert series

Assuming $(f_1, ..., f_k)$ is homogeneous, letting

$$deg(x_{i,(j)}) = j + 1$$

then $J_m(R)$ and $J_\infty(R)$ are also graded and we can define Hilbert-Poincaré series

$$H_q(J_\infty(R)) = \sum_{j\geq 0} \mathrm{dim}(J_\infty(R))_j q^j$$

Example

$R = k[x_1, ..., x_n]$. Then
$J_\infty(R) = k[x_{1,(0)}, x_{1,(1)}, ..., x_{2,(0)}, x_{2,(1)}, ..., x_{n,(0)}, x_{n,(1)}, ...]$.

Hilbert series

Assuming $(f_1, ..., f_k)$ is homogeneous, letting

$$deg(x_{i,(j)}) = j + 1$$

then $J_m(R)$ and $J_\infty(R)$ are also graded and we can define Hilbert-Poincaré series

$$H_q(J_\infty(R)) = \sum_{j\geq 0} \dim(J_\infty(R))_j q^j$$

Example

$R = k[x_1, ..., x_n]$. Then
$J_\infty(R) = k[x_{1,(0)}, x_{1,(1)}, ..., x_{2,(0)}, x_{2,(1)}, ..., x_{n,(0)}, x_{n,(1)}, ...]$.

$$H_q(J_\infty(R)) = \frac{1}{(q)_\infty^n}$$

h_Γ-series

h_Γ-series

Again, it is convenient to consider *two* representations

$$H_q(J_\infty(R)) = \frac{P_\Gamma(q)}{(q)_\infty^n} = \frac{h_\Gamma(q)}{(q)_\infty^k}$$

where k is the dimension of R.

h_Γ-series

Again, it is convenient to consider *two* representations

$$H_q(J_\infty(R)) = \frac{P_\Gamma(q)}{(q)_\infty^n} = \frac{h_\Gamma(q)}{(q)_\infty^k}$$

where k is the dimension of R.

Graph series and arc algebras

Let $\Gamma = (V, E)$ be a graph with no double edges and loops $\rightsquigarrow$ edge ideal:

$$R_\Gamma = \mathbb{C}[x_1, ..., x_n]/\langle x_i x_j : (i,j) \in E(\Gamma)\rangle.$$

Example

Node: $R = \mathbb{C}[x, y]/(xy)$. Then

$$J_\infty(R) = \mathbb{C}[x_0, x_1, ..., y_0, y_1, ...]/(x_0 y_0, x_1 y_0 + x_0 y_1, x_2 y_0 + 2x_1 y_1 + x_0 y_2, ...)$$

Graph series and arc algebras

Let $\Gamma = (V, E)$ be a graph with no double edges and loops $\rightsquigarrow$ edge ideal:

$$R_\Gamma = \mathbb{C}[x_1, ..., x_n]/\langle x_i x_j : (i,j) \in E(\Gamma)\rangle.$$

Example

Node: $R = \mathbb{C}[x, y]/(xy)$. Then

$$J_\infty(R) = \mathbb{C}[x_0, x_1, ..., y_0, y_1, ...]/(x_0y_0, x_1y_0+x_0y_1, x_2y_0+2x_1y_1+x_0y_2, ...)$$

$$H_q(J_\infty(R_\Gamma)) = \sum_{n_1,n_2\geq 0} \frac{q^{n_1+n_2+n_1n_2}}{(q)_{n_1}(q)_{n_2}} = \frac{\frac{1}{(1-q)}}{(q)_\infty}$$

An old result

A reformulation of our old result with M. Penn (2011,2012):

Theorem

For any graph Γ without multiple edges

$$H_\Gamma(q) = H_q(J_\infty(R_\Gamma)).$$

Moreover, this agrees with the character of a certain "principal" vertex algebra.

q-series identities from graph series

q-series identities from graph series

Many interesting identities. For instance, for path graphs A_n, $1 \leq n \leq 9$ we are able to simplify $H_{\Gamma_{A_n}}(q)$ up to a single summation.

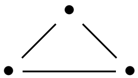
$$\bullet - - \bullet - - \cdots - - \bullet$$

Proposition

$$H_{A_7}(q) = \frac{\sum_{m\geq 1}(-3m+1)(-1)^m q^{\frac{3m^2+m}{2}} + \sum_{m\leq -1}(3m+2)(-1)^m q^{\frac{3m^2+m}{2}}}{(1-q)(q)_\infty^4}$$

5th order mock theta functions

5th order mock theta functions

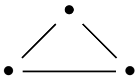


C_3

$$H_{C_3}(q) = \frac{1}{(q)_\infty} \underbrace{\sum_{n\geq 0} \frac{q^n}{(q^{n+1})_{n+1}}}_{=\chi_1(q)}$$

There is also formula for $\chi_0(q)$.

5th order mock theta functions

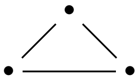


C_3

$$H_{C_3}(q) = \frac{1}{(q)_\infty} \underbrace{\sum_{n\geq 0} \frac{q^n}{(q^{n+1})_{n+1}}}_{=\chi_1(q)}$$

There is also formula for $\chi_0(q)$. By Zwegers (2009)

5th order mock theta functions



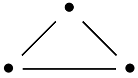
$$H_{C_3}(q) = \frac{1}{(q)_\infty} \underbrace{\sum_{n\geq 0} \frac{q^n}{(q^{n+1})_{n+1}}}_{=\chi_1(q)}$$

There is also formula for $\chi_0(q)$. By Zwegers (2009)

$$\frac{1}{(q)_\infty} \sum_{n\geq 0} \frac{q^n}{(q^{n+1})_{n+1}}$$

$$= \frac{1}{(q)_\infty^3} \left(\sum_{k,\ell,m\geq 0} - \sum_{k,\ell,m<0} \right) (-1)^{k+\ell+m} q^{\frac{1}{2}k^2+\frac{1}{2}\ell^2+\frac{1}{2}m^2+2k\ell+2\ell m+2km+\frac{3}{2}(k+\ell+m)}$$

5th order mock theta functions



$$H_{C_3}(q) = \frac{1}{(q)_\infty} \underbrace{\sum_{n\geq 0} \frac{q^n}{(q^{n+1})_{n+1}}}_{=\chi_1(q)}$$

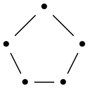
There is also formula for $\chi_0(q)$. By Zwegers (2009)

$$\frac{1}{(q)_\infty} \sum_{n\geq 0} \frac{q^n}{(q^{n+1})_{n+1}}$$

$$= \frac{1}{(q)_\infty^3} \left(\sum_{k,\ell,m\geq 0} - \sum_{k,\ell,m<0} \right) (-1)^{k+\ell+m} q^{\frac{1}{2}k^2+\frac{1}{2}\ell^2+\frac{1}{2}m^2+2k\ell+2\ell m+2km+\frac{3}{2}(k+\ell+m)}$$

With a PhD student we were able to interpret the RHS using algebra.

More complicated graphs



C_5

$$H_{C_5}(q) = \frac{q^{-1}}{(q)_\infty^2} \sum_{n\geq 1} \frac{nq^n}{1-q^n}$$

This is the first example in an infinite family of graphs with $3k+2$ vertices, $k \geq 1$ for which we can express h_Γ as the generating series of certain sums of power of divisors.

Further q-series identities: D series

Further q-series identities: D series

Theorem (Bringmann-Jennings-Shaffer-A.M.)

We have

$$H_{D_4}(q) = \frac{\sum_{n,m\geq 0}(-1)^{m+n}(2n+1)q^{\frac{3}{2}m^2+\frac{5}{2}m+\frac{1}{2}n^2+\frac{3}{2}n+2mn}}{(q)_\infty^4}$$

$$H_{D_5}(q) = \frac{\left(\sum_{n,m\geq 0} - \sum_{n,m<0}\right)(-1)^n(n+1)^2 q^{\frac{n^2+3n}{2}+3mn+3m^2+4m}}{(q)_\infty^5}$$

Both numerators are indefinite theta series of signature $(1,1)$*.*
They are both mixed mock modular forms.

Multiple edges

B_2 graph:

$$\bullet === \bullet$$

$$H_{B_2}(q) = \sum_{n_1, n_2 \geq 0} \frac{q^{n_1+n_2+2n_1n_2}}{(q)_{n_1}(q)_{n_2}}$$

Proposition

$$H_{B_2}(q) = \frac{1}{(q)_\infty} \sum_{n \geq 1} \chi(n) q^{\frac{n^2-49}{120}}$$

where $\chi(n) = (-1)^{[\frac{n}{30}]}$ *if* $n^2 \equiv 49 \mod 120$ *and zero otherwise.*

This is a famous q-series appearing in Lawrence-Zagier's work on WRT invariants of $\Sigma(2,3,5)$.

Modular properties of graph series

What kind of q-series can we get out of $q^a H_\Gamma(q)$?

- (mixed) quantum modular forms
- inside $\mathcal{QM} := \mathbb{Q}[E_2, E_4, E_6]$
- (mixed) mock theta functions
- modular? asymptotic behavior?

Modular properties of graph series

What kind of q-series can we get out of $q^a H_\Gamma(q)$?

- (mixed) quantum modular forms
- inside $\mathcal{QM} := \mathbb{Q}[E_2, E_4, E_6]$
- (mixed) mock theta functions
- modular? asymptotic behavior?

Example

Some graph series (modulo Euler products) whose modularity properties are unknown:

$$\sum_{n\geq 1} q^n (q)_n^3$$

$$\sum_{n,m\geq 1} q^{mn+m+n}(q)_m(q)_n$$

$$\sum_{n,m\geq 1} \frac{q^{mn}}{(q)_{m+n+1}}$$

Generalizations

Generalizations

Graphs with loops:
Single node with loops $\rightsquigarrow$ "fat" point $R = \mathbb{C}[x]/(x^n) \rightsquigarrow J_\infty(R) \rightsquigarrow$ Andrews-Gordon series:

Feigin-Stoyanovsky, Feigin-Frenkel 1993

Capparelli-Lepowsky-A.M. 2005., Bruschek-Mourtada-Schepers 2011

More complicated ideals (not coming from graphs): Very few examples are known

Heluani-van Ekeren 2018, Andrews-Heluani-van Ekeren 2021

Li 2020 Li. A.M 2020

Part II

4d/2d dualities and Schur's index

Physics:

4d/2d dualities and Schur's index

Physics:

4d/2d dualities and Schur's index

Physics:

4d $\mathcal{N} = 2$ QFT is connected with many important developments in mathematics. If QFT is SCFT $\rightsquigarrow$ superconformal index.

4d/2d dualities and Schur's index

Physics:

4d $\mathcal{N} = 2$ QFT is connected with many important developments in mathematics. If QFT is SCFT $\rightsquigarrow$ superconformal index.

Connection with q-series and vertex algebras:

4d $\mathcal{N} = 2$ SCFT $\rightsquigarrow$ superconformal index $\rightsquigarrow$ Schur's index $\mathcal{I}(q)$ $\overset{4d/2d}{\leftrightsquigarrow}$ character (Hilbert series) of a vertex algebra.

Beem-Lemost-Liendo-Peelaers-Rastelli-van Rees 2013

4d/2d dualities and Schur's index

Physics:

4d $\mathcal{N}=2$ QFT is connected with many important developments in mathematics. If QFT is SCFT $\rightsquigarrow$ superconformal index.

Connection with q-series and vertex algebras:

4d $\mathcal{N}=2$ SCFT $\rightsquigarrow$ superconformal index $\rightsquigarrow$ Schur's index $\mathcal{I}(q)$ $\overset{4d/2d}{\leftrightsquigarrow}$ character (Hilbert series) of a vertex algebra.

Beem-Lemost-Liendo-Peelaers-Rastelli-van Rees 2013

Quantum dilogarithm

Quantum dilogarithm

Physicists proposed computation of $\mathcal{I}(q)$ using *wall-crossing technology* (after Kontsevich and Soibelman 2010, and Ceccotti-Neitzke-Vafa 2009). This computation is based on **quantum dilogarithm**:

$$E_q(X_i) = \prod_{i \geq 1} (1 + q^{i-1/2} X_i)^{-1}$$

(here X_i are non-commutative variables!)

Conjecture: Very roughly speaking:
Quiver (oriented graph) $\Gamma \rightsquigarrow$ product of quantum dilogarithms $\rightsquigarrow$ constant term $\rightsquigarrow$ q-series representation for $\mathcal{I}(q)$

Cordova, Shao, Gaiotto 2016,2018

Toy case

- (single node and no edges). There is only one variable X so everything is commutative.

Toy case

• (single node and no edges). There is only one variable X so everything is commutative.
We have (after Ramanujan, Rogers,...)

$$\mathcal{I}_\Gamma(q) := \mathrm{CT}_X E_q(X)E_q(X^{-1}) = \mathrm{CT}_X \frac{1}{\prod_{n\geq 1}(1+Xq^{n-1/2})(1+X^{-1}q^{n-1/2})}$$

$$= \sum_{n\geq 0} \frac{q^n}{(q)_n^2} = \frac{\sum_{n\in\mathbb{Z}} \mathrm{sgn}(n) q^{2n^2+n}}{(q)_\infty^2}$$

Double graph series

For certain quivers same type of computation (with non-commutative variables!) gives

Definition (Graph series with "double poles")

Everything as before but with double poles

$$\sum_{n_1,\dots,n_r\geq 0} \frac{q^{n_1+\cdots+n_r+\frac{1}{2}\mathrm{n}C\mathrm{n}^T}}{(q)_{n_1}^2\cdots(q)_{n_r}^2},$$

where C is the adjacency matrix of the *underlying graph*. Up to Euler's factors this is supposed to agree with the Schur's index (or character) $\mathcal{I}(q)$.

Basic identity

Pentagon identity :
With $X_1X_2 = qX_2X_1$, we have

$$E_q(X_1)E_q(X_2) = E_q(X_2)E_q(X_1X_2)E_q(X_1)$$

Quiver theories

ADE quiver diagram with orientation: $\leftarrow$ and $\rightarrow$ (sink and sources).

"Non-commutative Jacobi form":

$$\prod_{J'\in Sou} E_q(X_{-\gamma_{J'}}) \prod_{I'\in Sink} E_q(X_{-\gamma_J}) \prod_{J\in Sou} E_q(X_{\gamma_J}) \prod_{I\in Sink} E_q(X_{\gamma_I})$$

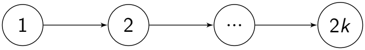
Quiver theories

ADE quiver diagram with orientation: $\leftarrow$ and $\rightarrow$ (sink and sources).

"Non-commutative Jacobi form":

$$\prod_{J'\in Sou} E_q(X_{-\gamma_{J'}}) \prod_{I'\in Sink} E_q(X_{-\gamma_J}) \prod_{J\in Sou} E_q(X_{\gamma_J}) \prod_{I\in Sink} E_q(X_{\gamma_I})$$

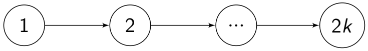
Quivers of type A_{2k}



It is known that the index $\mathcal{I}_{A_{2k}}(q)$ is given by

$$\prod_{\substack{i \geq 1 \\ i \not\equiv 0, \pm 1 \ (2k+3)}} \frac{1}{(1-q^i)}$$

Quivers of type A_{2k}

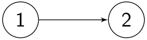


It is known that the index $\mathcal{I}_{A_{2k}}(q)$ is given by

$$\prod_{\substack{i\geq 1 \\ i\not\equiv 0,\pm 1 \ (2k+3)}} \frac{1}{(1-q^i)}$$

Famous product side in (one of) the Andrews-Gordon identities. In particular for $k=1$,

Quiver of type A_2: Rogers-Ramanujan series



Example

$$\mathcal{I}(q) \stackrel{?}{=} (q)_\infty^4 \mathrm{CT}[E_q(X_{-\gamma_1})E_q(X_{-\gamma_2})E_q(X_{\gamma_1})E_q(X_{\gamma_2})]$$
$$= (q)_\infty^4 \sum_{n_1,n_2\geq 0} \frac{q^{n_1+n_2+n_1 n_2}}{(q)_{n_1}^2 (q)_{n_2}^2}$$

It is not hard to see that the RHS is $\frac{1}{\prod_{n\geq 1}(1-q^{5n+2})(1-q^{5n+3})}$.

Quivers of type A_{2k}

Similar computation gives

$$\mathcal{I}_{A_{2k}}(q) \stackrel{?}{=} (q)_\infty^{2k} \sum_{n_1, n_2, \ldots, n_{2k} \geq 0} \frac{q^{\sum_{i=1}^{2k-1} n_i n_{i+1} + \sum_{i=1}^{2k} n_i}}{(q)_{n_1}^2 (q)_{n_2}^2 \cdots (q)_{n_{2k}}^2}$$

Cordova-Shao 2016

General case

Of course, physicists are always right.

Theorem

For $k \geq 1$,

$$\prod_{\substack{i\geq 1 \\ i\not\equiv 0,\pm 1 \ (2k+1)}} \frac{1}{(1-q^i)} = (q)_\infty^{2k} \sum_{n_1,n_2,\ldots,n_{2k}\geq 0} \frac{q^{\sum_{i=1}^{2k-1} n_i n_{i+1} + \sum_{i=1}^{2k} n_i}}{(q)_{n_1}^2 (q)_{n_2}^2 \cdots (q)_{n_{2k}}^2}$$

This is very different compared to Andrews-Gordon identities.

Quivers of A_{2k+1} type

Theorem (Jennings-Shaffer-A.M.)

For $k \geq 1$,

$$\frac{\sum_{n\in\mathbb{Z}} \operatorname{sgn}(n) q^{(k+1)n^2+kn}}{(q)_\infty} = (q)_\infty^{2k-1} \sum_{n_1,n_2,\ldots,n_{2k-1}\geq 0} \frac{q^{\sum_{i=1}^{2k-2} n_i n_{i+1} + \sum_{i=1}^{2k-1} n_i}}{(q)_{n_1}^2 (q)_{n_2}^2 \cdots (q)_{n_{2k-1}}^2}$$

Quivers of A_{2k+1} type

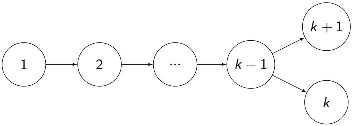
Theorem (Jennings-Shaffer-A.M.)

For $k \geq 1$,

$$\frac{\sum_{n\in\mathbb{Z}} \operatorname{sgn}(n) q^{(k+1)n^2+kn}}{(q)_\infty} = (q)_\infty^{2k-1} \sum_{n_1,n_2,\dots,n_{2k-1}\geq 0} \frac{q^{\sum_{i=1}^{2k-2} n_i n_{i+1} + \sum_{i=1}^{2k-1} n_i}}{(q)_{n_1}^2 (q)_{n_2}^2 \cdots (q)_{n_{2k-1}}^2}$$

For $k = 1$ this gives Ramanujan's formula discussed earlier.

Quivers of D type



The relevant double pole q-series is:

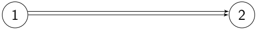
$$\sum_{n_1,n_2,\ldots,n_{k+1}\geq 0} \frac{q^{\sum_{i=1}^{k-1} n_i n_{i+1} + n_{k-1} n_{k+1} + \sum_{i=1}^{k+1} n_i}}{(q)_{n_1}^2 (q)_{n_2}^2 \cdots (q)_{n_{k+1}}^2}.$$

This again alternates between modular and rank two false theta series (with some extra Euler factors).

Multiple edges

Multiple edges

Quivers with multiple edges, e.g



$$\sum_{n_1,n_2\geq 0} \frac{q^{n_1+n_2+2n_1n_2}}{(q)_{n_1}^2(q)_{n_2}^2} = \frac{\sum_{n\geq 0} q^{n^2+n}}{(q)_\infty^2}$$

Half-characteristic theta q-series

Half-characteristic theta q-series

New examples:

Theorem (Jennings-Shaffer-A.M.)

For $k \geq 2$,

$$(q)_\infty^k \sum_{n_1,n_2,\ldots,n_k \geq 0} \frac{q^{n_1 n_2 + n_2 n_3 + \cdots + n_{k-1} n_k + n_1 + n_2 + \cdots + n_k} (-q^{\frac{1}{2}})_{n_1}}{(q)_{n_1}^2 (q)_{n_2}^2 \cdots (q)_{n_k}^2}$$

$$= \frac{(-q^{\frac{1}{2}})_\infty}{(q)_\infty} \left(\sum_{n \geq 0} + (-1)^k \sum_{n<0} \right) (-1)^{(k+1)n} q^{\frac{(k+2)n^2 + (k+1)n}{2}}.$$

This again alternates between false and modular identities (essentially Andrews-Bressoud series).

What about other ABG-type series?

What about other ABG-type series?

There are double pole identities for *all* AB and AG series and *all* related false theta series, but formulas are more complicated. For instance, for AG series

What about other ABG-type series?

There are double pole identities for *all* AB and AG series and *all* related false theta series, but formulas are more complicated. For instance, for AG series

Theorem (Kanade, A.M., Russell)

For $k \geq 1$, and $1 \leq i \leq k$

$$\prod_{\substack{n \geq 1 \\ n \not\equiv 0, \pm i \ (2k+1)}} \frac{1}{(1-q^i)} = (q)_\infty^{2k} \sum_{n_1, n_2, \ldots, n_{2k} \geq 0} \frac{a_i(q) q^{\sum_{i=1}^{2k-1} n_i n_{i+1} + \sum_{i=1}^{2k} n_i}}{(q)_{n_1}^2 (q)_{n_2}^2 \cdots (q)_{n_{2k}}^2}$$

where

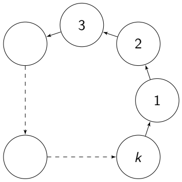
$$a_1 = 1, a_2 = 2 - q^{n_1}, a_3 = 2 - 2q^{n_1} + q^{n_2}, \ldots$$

In the simplest case this was conjectured by Cordova, Gaiotto and Shao.

Circles, Triangles and Squares...

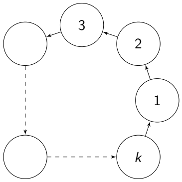
Circles, Triangles and Squares...

k-cycle quiver ($k \geq 3$):



Circles, Triangles and Squares...

k-cycle quiver ($k \geq 3$):



Conjecture

For $k \geq 3$,

$$\frac{\sum_{n\geq 0}(-1)^{nk}q^{\frac{k}{2}n(n+1)}}{(q)_\infty^k} = \sum_{n_1,n_2,\ldots,n_k\geq 0} \frac{q^{\sum_{i=1}^{k-1} n_i n_{i+1} + n_k n_1 + \sum_{i=1}^{k} n_i}}{(q)_{n_1}^2 (q)_{n_2}^2 \cdots (q)_{n_k}^2},$$

Part III

q-MZVs

q-MZVs

In its "standard" form, the q-MZV is usually defined as

$$\zeta_q(a_1, ..., a_k) := \sum_{n_1 > n_2 > \cdots > n_k \geq 1} \frac{q^{(a_1-1)n_1 + \cdots + (a_k-1)n_k}}{(1-q^{n_1})^{a_1} \cdots (1-q^{n_k})^{a_k}},$$

where $a_i \in \mathbb{N}$ and $a_1 \geq 2$.

q-MZVs

In its "standard" form, the q-MZV is usually defined as

$$\zeta_q(a_1, ..., a_k) := \sum_{n_1 > n_2 > \cdots > n_k \geq 1} \frac{q^{(a_1-1)n_1 + \cdots + (a_k-1)n_k}}{(1-q^{n_1})^{a_1} \cdots (1-q^{n_k})^{a_k}},$$

where $a_i \in \mathbb{N}$ and $a_1 \geq 2$.

$$\zeta_q^*(a_1, ..., a_k) := \sum_{n_1 \geq n_2 \geq \cdots \geq n_k \geq 1} \frac{q^{(a_1-1)n_1 + \cdots + (a_k-1)n_k}}{(1-q^{n_1})^{a_1} \cdots (1-q^{n_k})^{a_k}},$$

The star symbol indicates that the summation is over non-strict summation variables.

Another model of q-MZVs

$$\mathfrak{z}_q(a_1, \ldots, a_k) := \sum_{n_1 > n_2 > \cdots > n_k \geq 1} \frac{q^{n_1}}{(1-q^{n_1})^{a_1} \cdots (1-q^{n_k})^{a_k}}.$$

$$\mathfrak{z}^*_q(a_1, \ldots, a_k) := \sum_{n_1 \geq n_2 \geq \cdots \geq n_k \geq 1} \frac{q^{n_1}}{(1-q^{n_1})^{a_1} \cdots (1-q^{n_k})^{a_k}}.$$

Very active area of research.

Bradley, Hoffman, Zhao, Schlesinger, Okounkov, Zudilin, Ohno,...

Another model of q-MZVs

$$\mathfrak{z}_q(a_1, \ldots, a_k) := \sum_{n_1 > n_2 > \cdots > n_k \geq 1} \frac{q^{n_1}}{(1-q^{n_1})^{a_1} \cdots (1-q^{n_k})^{a_k}}.$$

$$\mathfrak{z}^*_q(a_1, \ldots, a_k) := \sum_{n_1 \geq n_2 \geq \cdots \geq n_k \geq 1} \frac{q^{n_1}}{(1-q^{n_1})^{a_1} \cdots (1-q^{n_k})^{a_k}}.$$

Very active area of research.

Bradley, Hoffman, Zhao, Schlesinger, Okounkov, Zudilin, Ohno,...

$\lim_{q \to 1-}$ "recovers" $\zeta(a_1, \ldots, a_k)$

Graphs series and q-MZVs

Graphs series and q-MZVs

Theorem (A.M.)

For every choice of positive integers $a_1, \ldots, a_k$ there is a simple graph $Z_{a_1,\ldots,a_k}$ such that

$$H_{Z_{a_1,\ldots,a_k}}(q) = \frac{q^{-1}\mathfrak{z}_q^*(a_1, \ldots, a_k)}{(q)_\infty^{k+a_1+\cdots+a_k}}.$$

One can also engineer graph series involving certain generalized q-MZV type sums called *brackets*.

Bachmann-Kühn

q-MZVs associated to simple Lie algebras

q-MZVs associated to simple Lie algebras

Denote by Δ a root system of ADE type (for simplicity), Δ_+ the set of positive roots and $\langle \cdot, \cdot \rangle$ denotes inner product normalized such that $\langle \alpha, \alpha \rangle = 2$ for every root α. Then we let for $k_\alpha \geq 1$,

$$\zeta_{\mathfrak{g},q}(k_1, .., k_{|\Delta_+|}) := \sum_{\lambda \in P_+} \frac{q^{\frac{1}{2}\sum_{\alpha\in\Delta_+} k_\alpha \langle \lambda+\rho, \alpha\rangle}}{\prod_{\alpha\in\Delta_+}(1-q^{\langle \lambda, \alpha+\rho\rangle})^{k_\alpha}},$$

where the summation is over the cone of positive dominant integral weights.

Example

For $\mathfrak{g} = \mathfrak{sl}_2$ and $\mathfrak{g} = \mathfrak{sl}_3$, and $k \geq 2$,

$$\sum_{n\geq 1} \frac{q^{\frac{k}{2}n}}{(1-q^n)^k}$$

$$\sum_{n_1,n_2\geq 1} \frac{q^{\frac{k_1}{2}n_1 + \frac{k_2}{2}n_2 + \frac{k_3}{2}(n_1+n_2)}}{(1-q^{n_1})^{k_1}(1-q^{n_2})^{k_2}(1-q^{n_1+n_2})^{k_3}}$$

q-MZVs and quasi-modularity

In parallel with standard q-MZVs, we expect

Conjecture

$$\zeta_{\mathfrak{g},q}(2k) := \zeta_{\mathfrak{g},q}(2k, 2k, ..., 2k) \in \mathbb{Q}[E_2, E_4, E_6].$$

q-MZVs and quasi-modularity

In parallel with standard q-MZVs, we expect

Conjecture

$$\zeta_{\mathfrak{g},q}(2k) := \zeta_{\mathfrak{g},q}(2k, 2k, ..., 2k) \in \mathbb{Q}[E_2, E_4, E_6].$$

A closely related q-series appeared recently in connection to Schur's indices:

$$\mathcal{I}_{\mathfrak{g},k}(q) := \sum_{\lambda \in P_+} P_k(\lambda) \frac{q^{\frac{1}{2}\sum_{\alpha\in\Delta_+} k\langle\lambda+\rho,\alpha\rangle}}{\prod_{\alpha\in\Delta_+}(1-q^{\langle\lambda,\alpha+\rho\rangle})^k},$$

It is expected that for k even $\mathcal{I}_{\mathfrak{g},k}(q) \in \mathcal{QM}$.

Beem-Rastelli 2018, Arakawa 2018, A.M. 2022

This is known in many special cases.

Thank You!