Asymptotics of Restricted Partition Functions

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Integer Partitions

A partition of a number n is a non-increasing sequence of positive integers whose sum is equal to n:

$$n = a_1 + a_2 + \cdots + a_m, \quad a_i \geq a_{i+1} > 0.$$

The number of partitions of n is denoted $p(n)$.
Example: $p(4) = 5$:

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Theorem (Hardy-Ramanujan, 1918)

$$p(n) \sim \frac{1}{4\sqrt{3}} \exp\left(\pi\sqrt{\frac{2}{3}} n^{1/2}\right) n^{-1}$$

Restricted Partitions

Let $\mathcal{A} \subset \mathbb{N}$ with $\gcd(\mathcal{A}) = 1$.
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1934 – 1969: Wright, Bateman-Erdős, Browkin, Roth-Szekeres, Kerawala

- Various results for very general $\mathcal{A}$
- Complicated formulas, long and difficult proof methods

Modern examples of $p_{\mathcal{A}}(n)$

- Perfect k^{th} powers:

$$\mathcal{A}_k = \{x^k : x \in \mathbb{N}\}$$

Example: $p_{\mathcal{A}_2}(10) = 4$:

$$9+1, \quad 4+4+1+1, \quad 4+1+\cdots+1, \quad 1+\cdots+1$$

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Theorem (G. 2016)

$$p_{\mathcal{A}_k}(n) \sim C_1 \exp\left(C_2 n^{\frac{1}{k+1}}\right) n^{-\frac{3k+1}{2(k+1)}}$$

where C_1, C_2 are constants depending only on k.

The case $k = 2$ is due to Vaughan, 2014.

Modern examples of $p_{\mathcal{A}}(n)$

- Perfect k^{th} Powers in a residue class:

$$\mathcal{A}_{k,(a,b)} = \{x^k : x \equiv a \ (\mathrm{mod}\, b), x \in \mathbb{N}\}$$

Theorem (Berndt-Malik-Zaharescu, 2018)

$$p_{\mathcal{A}_{k,(a,b)}}(n) \sim C_1 \exp\left(C_2 n^{\frac{1}{k+1}}\right) n^{-\frac{b+bk+2ak}{2b(k+1)}}$$

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- Values of a polynomial: Let f be a suitable polynomial such that $f(\mathbb{N}) \subset \mathbb{N}$.

$$\mathcal{A}_f = \{f(x) : x \in \mathbb{N}\}$$

Theorem (Dunn-Robles, 2018)

$$p_{\mathcal{A}_f}(n) \sim C_1 \exp\left(C_2 n^{\frac{1}{d+1}}\right) n^{-\frac{2d(1-\zeta(0,\alpha))+1}{2(d+1)}}$$

where $d = \deg(f)$, C_1, C_2 are constants depending only on f, and $\zeta(0, \alpha)$ is a value of an appropriate Matsumoto-Weng ζ function.

Modern examples of $p_{\mathcal{A}}(n)$

- Primes:

$$\mathcal{A} = \mathbb{P} = \{\text{primes}\}$$

Theorem (Vaughan, 2007)

$$p_{\mathbb{P}}(n) \sim C_1 \exp\left(C_2\,\pi(n)^{\frac{1}{2}}\right) n^{-\frac{3}{4}} (\log n)^{-\frac{1}{4}}$$

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- Powers of Primes:

$$\mathcal{A} = \mathbb{P}_k = \{p^k : p \text{ prime}\}$$

Theorem (G., 2021)

$$p_{\mathbb{P}_k}(n) \sim C_1 \exp\left(C_2\, \pi(n^{\frac{1}{k}})^{\frac{k}{k+1}}\right) n^{-\frac{(2k+1)k}{(k+1)^2}} (\log n)^{-\frac{k^2}{(k+1)^2}}$$

where C_1, C_2 are constants depending only on k.

Sketch of the proofs

The partition function $p(n)$ has generating function

$$\Psi(z) = \sum_{n=0}^{\infty} p(n) z^n = \prod_{m=1}^{\infty} \frac{1}{1 - z^m},$$

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$$= (1 + z^1 + z^{1+1} + \cdots)(1 + z^2 + z^{2+2} + \cdots)(1 + z^3 + z^{3+3} + \cdots) \cdots$$

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Restricting parts to the set $\mathcal{A}$, we get a generating function for $p_{\mathcal{A}}(n)$:

$$\Psi_{\mathcal{A}}(z) = \sum_{n=0}^{\infty} p_{\mathcal{A}}(n) z^n = \prod_{a \in \mathcal{A}} \frac{1}{1 - z^a}.$$

Hardy-Littlewood Circle Method

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$$p_{\mathcal{A}}(n) = \int_0^1 \Psi_{\mathcal{A}}(\rho e(\alpha)) \rho^{-n} e(-n\alpha) \, d\alpha.$$

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It is often more convenient to use

$$\Phi_{\mathcal{A}}(z) = \sum_{j=1}^{\infty} \sum_{a \in \mathcal{A}} \frac{1}{j} z^{ja},$$

so that

$$\Psi_{\mathcal{A}}(z) = \exp(\Phi_{\mathcal{A}}(z)).$$

Major and Minor Arcs

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The gaps between the major arcs are called Minor Arcs, and are absorbed into the error term.
We divide the integral into 3 parts:

$$\underbrace{\int_{\mathfrak{M}(1,0)}}_{\uparrow\ \alpha \text{ close to } 0} + \underbrace{\sum_{1\le a<q\le Q} \int_{\mathfrak{M}(q,a)}}_{\uparrow\ \alpha \text{ close to } \frac{a}{q}} + \underbrace{\int_{\mathfrak{m}}}_{\uparrow\ \text{the rest}}$$

The principal major arc $\mathfrak{M}(1,0)$

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$$\Phi(\rho) = \frac{1}{2\pi i} \sum_{j=1}^{\infty} \sum_{a\in\mathcal{A}} \frac{1}{j} \int_{c-i\infty}^{c+i\infty} \Gamma(s) j^{-s} a^{-s} X^{s}\, ds.$$

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Bringing the sum over $\mathcal{A}$ inside the integral, we see that we need to study sums of the form

$$\sum_{a \in \mathcal{A}} a^{-s}.$$

We need analytic information such as zeros, poles, residues, etc.

Special ζ-functions

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- For polynomial values: Matsumoto-Weng ζ function
$$\sum_{n=1}^{\infty} f(n)^{-s} = \zeta_f(s).$$
- For primes and prime powers: Prime ζ function
$$\sum_{p \text{ prime}} p^{-s} = P(s); \quad \sum_{p \text{ prime}} p^{-ks} = P(ks).$$

The contour integral

We need a good estimate for

$$\Phi(\rho) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \zeta(s+1)\Gamma(s) \left(\sum_{a \in \mathcal{A}} a^{-s} \right) X^s \, ds.$$

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We need to stay in the zero-free region of ζ, so we use a keyhole contour to integrate around the singularity at $s = \frac{1}{k}$.

The non-principal major arcs

Now we need a good estimate for

$$\Phi\left(\rho e\left(\frac{r}{q}+\beta\right)\right)=\sum_{j=1}^{\infty}\frac{1}{j}\sum_{a\in\mathcal{A}}e\left(\frac{ajr}{q}\right)\exp\left(aj(2\pi i\beta-1/X)\right).$$

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Splitting the sum according to residue classes, this becomes

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To bound this, we need to understand the distribution of $\mathcal{A}$ in residue classes mod q.

Distribution of $\mathcal{A}$ in residue classes

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- For k^{th} powers or polynomial values:
  - Distribution comes from solving the polynomial mod q
  - Can take denominators up to a power of X
  - Need to analyze

$$S(q, a) = \sum_{\ell=1}^{q} e\left(\frac{r\ell^k}{q}\right)$$

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- For primes and prime powers:
  - Distribution given by Siegel-Walfisz theorem
  - Only valid when $q \leq (\log X)^B$
  - Need to analyze

$$S^*(q,a) = \sum_{\substack{\ell=1 \\ (\ell,q)=1}}^{q} e\left(\frac{r\ell^k}{q}\right)$$

The goal in both cases is to save a uniform constant $(1-\delta)$ factor over the trivial bound (q or $\varphi(q)$).

Bounding the non-principal major arcs

Recall that the principal major arc yields a main term of the form:

$$C_1 \exp\left(C_2\, \pi (n^{\frac{1}{k}})^{\frac{k}{k+1}}\right) n^{-\frac{(2k+1)k}{(k+1)^2}} (\log n)^{-\frac{k^2}{(k+1)^2}}.$$

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By saving a uniform constant factor in $S(q,a)$ or $S^*(q,a)$, we find that the non-principal major arcs contribution is

$$\ll \exp\left((1-\delta)C_2\,\pi(n^{\frac{1}{k}})^{\frac{k}{k+1}}\right) n^{-\frac{(2k+1)k}{(k+1)^2}} (\log n)^{-\frac{k^2}{(k+1)^2}}$$
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In all cases, the only main term contribution comes from $\mathfrak{M}(1,0)$.

The minor arcs

Finally, we need to estimate

$$\Phi(\rho e(\alpha)) = \sum_{j=1}^{\infty} \sum_{a \in \mathcal{A}} \frac{1}{j} e^{-aj/X} e(ja\alpha)$$
$$= \sum_{j=1}^{\infty} \frac{1}{j} \int_0^{\infty} jX^{-1} e^{-jx/X} \sum_{\substack{a \leq x \\ a \in \mathcal{A}}} e(ja\alpha)\, dx$$

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So we need a good estimate for

$$\sum_{\substack{a \le x \\ a \in \mathcal{A}}} e(ja\alpha).$$

Special exponential sums

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Can we extend this to more general sets $\mathcal{A}$?

Three necessary ingredients:

$$\int_{\mathfrak{M}(1,0)} \quad + \quad \sum_{1\le a<q\le Q} \int_{\mathfrak{M}(q,a)} \quad + \quad \int_{\mathfrak{m}}$$

$\uparrow$	$\uparrow$	$\uparrow$
α close to 0	α close to $\frac{a}{q}$	the rest
ζ-function $\sum_{a\in\mathcal{A}} a^{-s}$	distribution of $\mathcal{A}$ in residue classes	Weyl sums $\sum_{\substack{a\le x \\ a\in\mathcal{A}}} e(a\theta)$.

Thank You!