Integers represented by positive-definite quaternary quadratic forms and Petersson inner products

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Motivating question

- Suppose that

$$Q(\vec{x}) = \sum_{i=1}^{r} \sum_{j=i}^{r} a_{ij} x_i x_j$$

is a positive-definite quadratic form with $a_{ij} \in \mathbb{Z}$ for all i, j.

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Theorem (Legendre, 1798)

If n is a positive integer, there are $x, y, z \in \mathbb{Z}$ with $n = x^2 + y^2 + z^2$ if and only if $n \neq 4^t(8k+7)$ for $t, k \geq 0$.

Necessary conditions

- A positive integer n is said to be *locally represented* by Q if there is a solution to $Q(\vec{x}) = n$ with $\vec{x} \in \mathbb{Z}_p^r$ for every p.

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Theorem (Tartakowski)

If $r \geq 5$, then a positive-definite form Q represents every sufficiently large locally represented positive integer n.

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- If $Q(x,y,z,w) \equiv 0 \pmod{49}$, then $x \equiv y \equiv z \equiv w \equiv 0 \pmod{7}$.

- It follows that Q doesn't represent $3 \cdot 7^k$ or $6 \cdot 7^k$ for any $k \geq 0$.

Anisotropic primes (2/2)

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Theorem (Tartakowski)

If Q has four variables, there are only finitely many anisotropic primes. Fix a positive integer m. Then there is a constant $C(Q, m)$ so that if n is locally represented, $\mathrm{ord}_p(n) \leq m$ for all anisotropic primes p, and $n > C(Q, m)$, then n is represented by Q.

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Theorem (Schulze-Pillot, 2001)

If n is coprime to any anisotropic prime, n is locally represented by Q, and $n \gg N(Q)^{14+\epsilon}$, then n is represented.

Prior work (2/3)

- We say that n satisfies the strong local solubility condition if for all primes p there is some $\vec{x} \in (\mathbb{Z}/p^r\mathbb{Z})^4$ so that $Q(\vec{x}) \equiv n \pmod{p^r}$ with $p \nmid A\vec{x}$. (We have $r = 3$ if $p = 2$ and $r = 1$ if $p > 2$.)

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• Browning and Dietmann's result is stronger when the successive minima of Q are close in size.

Main results (1/3)

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• The above result is not effective. It depends on zero-free regions for $GL(1)$ L-functions.

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Theorem (R, 2019)

Let Q be a quaternary quadratic form and suppose that n is locally represented by Q. If $\gcd(n, D(Q)) = 1$, then n is represented by Q provided $n \gg \max\{N(Q)^{3/2+\epsilon} D(Q)^{5/4+\epsilon}, N(Q)^{2+\epsilon} D(Q)^{1+\epsilon}\}$.

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$$n \gg \max\{N(Q)^{5/2+\epsilon} D(Q)^{9/4+\epsilon}, N(Q)^{3+\epsilon} D(Q)^{2+\epsilon}\}.$$

- If n is locally represented by Q, not represented, and

$$n \gg \max\{N(Q)^{9/2+\epsilon} D(Q)^{5/4+\epsilon}, N(Q)^{5+\epsilon} D(Q)^{1+\epsilon}\},$$

then there is an anisotropic prime p so that $p^2 | n$ and np^{2k} is not represented by Q for any $k \geq 1$.

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Theorem (Wu, 2020)

Let $S = \{Q_1, Q_2, \ldots, Q_s\}$ be any finite set of primitive positive-definite quaternary forms. Then there is a quaternary form R and an integer n so that $r_R(n) > r_{Q_i}(n)$ for all $1 \leq i \leq s$.

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Definitions

- A modular form of weight k, level N and character χ is a holomorphic function $f : \mathbb{H} \to \mathbb{C}$ so that

$$f\left(\frac{az+b}{cz+d}\right) = \chi(d)(cz+d)^k f(z)$$

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• Let $M_k(\Gamma_0(N), \chi)$ denote the $\mathbb{C}$-vector space of such modular forms, and $S_k(\Gamma_0(N), \chi)$ the subspace of cusp forms.

• These vector spaces are finite-dimensional!

Theta series

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$$\theta_Q(z) = \sum_{n=0}^{\infty} r_Q(n) q^n, \quad q = e^{2\pi i z}.$$

- The generating function $\theta_Q(z)$ is a modular form of weight 2 on $\Gamma_0(D(Q))$ with character $\chi_{D(Q)}$.

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- The coefficients of $a_C(n)$ are small and mysterious ($|a_C(n)| \ll d(n)\sqrt{n}$).

Example (1/2)

- If $Q = x^2 + y^2 + 3z^2 + 3w^2 + xz + yw$, then

$$\theta_Q(z) = 1 + 4q + 4q^2 + 8q^3 + 20q^4 + 16q^5 + \cdots \in M_2(\Gamma_0(11), \chi_1).$$

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- If

$$f(z) = q\prod_{n=1}^{\infty}(1-q^n)^2(1-q^{11n})^2 = \sum_{n=1}^{\infty} a(n)q^n,$$

then $C(z) = \frac{8}{5}f(z)$.

Example (2/2)

- The Hasse bound gives that $|a(n)| \leq d(n)\sqrt{n}$ and so

$$r_Q(n) \geq \frac{12}{5} \sum_{\substack{d|n \\ 11\nmid d}} d - \frac{8}{5} d(n)\sqrt{n}.$$

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- There are 110 squarefree integers for which the right hand side is negative.

- One can check that Q represents all of these. It follows that Q represents all positive integers.

Eisenstein part

- The coefficient $a_E(n)$ of the Eisenstein series can be written

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- We have $\beta_\infty(n) = \frac{\pi^2 n}{\sqrt{D(Q)}}$. If $p \nmid nD(Q)$, then $\beta_p(Q, n) = 1 + O(1/p^2)$. If $p | n$ but $p \nmid D(Q)$, then $\beta_p(Q) = 1 + O(1/p)$.

Bounds on $\beta_p(n)$

- Let p be a prime and decompose Q over $\mathbb{Z}_p$ as

$$p^{a_1} Q_1 \perp p^{a_2} Q_2 \perp \cdots \perp p^{a_k} Q_k.$$

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$$r_p(Q) = \min_{1 \leq i \leq k} \inf_{\substack{\vec{x} \in \mathbb{Z}_p^r \\ Q(\vec{x})=0}} \mathrm{ord}_p(a_i) + \mathrm{ord}_p(\vec{x}_i).$$

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- The $r_p(Q)$ is a measure of how anisotropic Q is. If Q is anisotropic, then $r_p(Q) = \infty$.

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- *In general, $\beta_p(n) \geq (1 - 1/p)p^{-\min\{r_p(Q), \mathrm{ord}_p(n)\}}$.*

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- *In general, $\beta_p(n) \geq (1 - 1/p)p^{-\min\{r_p(Q), \mathrm{ord}_p(n)\}}$.*

- We have similar results if $p = 2$.

The cusp form piece

- To bound $a_C(n)$, we use the same approach as the work of Fomenko and Schulze-Pillot. This is to bound

$$\langle C, C \rangle = \frac{3}{\pi[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N(Q))]} \iint_{\mathbb{H}/\Gamma_0(N)} |C(z)|^2 \, dx \, dy.$$

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- Blomer and Milićević show that there is an orthonormal basis $h_i = \sum a_i(n) q^n$ for $S_2(\Gamma_0(N(Q)), \chi)$ so that

$$a_i(n) \ll N(Q)^{1/2+\epsilon} d(n) \sqrt{n}$$

provided $\gcd(n, D(Q)) = 1$.

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- Let L be the lattice attached to Q. This is the set $\mathbb{Z}^4$ with the inner product

$$\langle \vec{x}, \vec{y} \rangle = \frac{1}{2} \left(Q(\vec{x} + \vec{y}) - Q(\vec{x}) - Q(\vec{y}) \right).$$

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- The dual lattice L' of L is $L' = \{\vec{x} \in \mathbb{R}^4 : \langle \vec{x}, \vec{y} \rangle \in \mathbb{Z} \text{ for all } \vec{y} \in L\}$.

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- For a number $c|N(Q)$, define D_c to be the kernel of the map $[c] : D \to D$ and D^c to be the image. Define

$$D^{c*} = \{\alpha \in D : \frac{1}{2}c\langle\gamma,\gamma\rangle + \langle\alpha,\gamma\rangle \equiv 0 \pmod 1 \text{ for all } \gamma \in D_c\}.$$

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- Let $w = \frac{c}{\gcd(N(Q),c)}$.

Formula

- With all the notation on the previous slide, the coefficient of $q^{n/w}$ in the Fourier expansion of $(cz+d)^{-2}\theta_Q((az+b)/(cz+d))$ is a root of unity times

$$\frac{1}{\sqrt{|D^{c*}|}} \sum_{\beta \in D^{c*}} e^{\pi i a \langle \beta, \beta \rangle /2} \#\{\vec{v} \in \bigcup L + \beta : \beta \in D^{c*}, Q(\vec{v}) = n/w\}.$$

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- Let $T = \{\vec{x} \in L' : \vec{x} \bmod L \subseteq D^c \cup D^{c*}\}$.

- If we define $R : T \to \mathbb{Q}$ by

$$R(\vec{x}) = 4w\langle \vec{x}, \vec{x} \rangle,$$

then R is an integral quadratic form with discriminant $\leq \frac{(4w)^4 D(Q)}{|D^c|^2}$.

Bound on $\langle C, C \rangle$

- Putting this together, we get

$$\langle C, C \rangle \ll \frac{1}{[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N)]} \sum_{a/c} w \sum_{n=1}^{\infty} \frac{r_R(4n)^2}{|D^{c*}|(n/w)} e^{-2\pi\sqrt{3}n/w}.$$

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- To estimate the sum, we need to bound $\sum_{n \leq x} r_R(n)^2$.

Analyzing this sum

- The easiest way to do this is to use

$$\sum_{n \leq x} r_R(n)^2 \leq \left(\sum_{n \leq x} r_R(n) \right) \cdot \left(\max_{n \leq x} r_R(n) \right).$$

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- There's a clever argument I learned from MathOverflow that gives

$$\max_{n \leq x} r_R(n) \ll x^{1+\epsilon} D(R)^{-1/4+\epsilon} + x^{1/2}.$$

Cusp form bound

- From this we get that

$$\langle C, C \rangle \ll \frac{1}{[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N(Q))]} \sum_{a/c} \frac{w^3}{|D^{c*}|}.$$

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- It follows from this that $|a_C(n)| \ll E(Q) d(n) \sqrt{n}$ if $\gcd(n, D(Q)) = 1$.

Conclusion

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- If $\gcd(n, D(Q)) = 1$, then $|a_C(n)| \ll E(Q) d(n) \sqrt{n}$.
- It follows that $r_Q(n) > 0$ if $n \gg D(Q) E(Q)^{2+\epsilon}$.

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Theorem (Lagrange, 1770)

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- Write $Q(\vec{x}) = \frac{1}{2}\vec{x}^T A\vec{x}$. We say that Q is *integer-matrix* if all the entries of A are even.

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Theorem (Conway-Schneeberger-Bhargava)

A positive-definite integer matrix form Q represents every positive integer if and only if it represents 1, 2, 3, 5, 6, 7, 10, 14, *and* 15.

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Theorem (Bhargava-Hanke)

A positive-definite, integer-valued form Q represents every positive integer if and only if it represents

$$1, 2, 3, 5, 6, 7, 10, 13, 14, 15, 17, 19, 21, 22, 23, 26, 29,$$
$$30, 31, 34, 35, 37, 42, 58, 93, 110, 145, 203, \textit{and } 290.$$

Consequences

- Each of these results is sharp. The form

$$x^2 + 2y^2 + 4z^2 + 29w^2 + 145v^2 - xz - yz$$

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- If a form represents every positive integer less than 290, it represents every integer greater than 290.

- There are 6436 integer-valued quaternary forms that represent all positive integers.

Later results

Theorem (R, 2014)

Assume GRH. Then a positive-definite, integer-valued form Q represents all positive odds if and only if it represents

$$1, 3, 5, 7, 11, 13, 15, 17, 19, 21, 23, 29, 31, 33, 35, 37, 39, 41, 47,$$
$$51, 53, 57, 59, 77, 83, 85, 87, 89, 91, 93, 105, 119, 123, 133, 137,$$
$$143, 145, 187, 195, 203, 205, 209, 231, 319, 385, \textit{and } 451.$$

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Theorem (DeBenedetto-R, 2016)

A positive-definite, integer-valued form Q represents every positive integer coprime to 3 if and only if it represents

$$1, 2, 5, 7, 10, 11, 13, 14, 17, 19, 22, 23, 26, 29, 31, 34, 35$$
$$37, 38, 46, 47, 55, 58, 62, 70, 94, 110, 119, 145, 203, \textit{and } 290.$$

Two exceptions

- It follows from the proof of the 15-theorem that if an integer-valued form Q represents all positive integers with one exception, then that exception must be 1, 2, 3, 5, 6, 7, 10, 14, or 15.

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Theorem (BDMSST, 2017)

If a positive-definite integer-matrix form Q represents all positive integers with two exceptions, the pair of exceptions $\{m, n\}$ must be one of the following: $\{1,2\}$, $\{1,3\}$, $\{1,4\}$, $\{1,5\}$, $\{1,6\}$, $\{1,7\}$, $\{1,9\}$, $\{1,10\}$, $\{1,11\}$, $\{1,13\}$, $\{1,14\}$, $\{1,15\}$, $\{1,17\}$, $\{1,19\}$, $\{1,21\}$, $\{1,23\}$, $\{1,25\}$, $\{1,30\}$, $\{1,41\}$, $\{1,55\}$, $\{2,3\}$, $\{2,5\}$, $\{2,6\}$, $\{2,8\}$, $\{2,10\}$, $\{2,11\}$, $\{2,14\}$, $\{2,15\}$, $\{2,18\}$, $\{2,22\}$, $\{2,30\}$, $\{2,38\}$, $\{2,50\}$, $\{3,6\}$, $\{3,7\}$, $\{3,11\}$, $\{3,12\}$, $\{3,19\}$, $\{3,21\}$, $\{3,27\}$, $\{3,30\}$, $\{3,35\}$, $\{3,39\}$, $\{5,7\}$, $\{5,10\}$, $\{5,13\}$, $\{5,14\}$, $\{5,20\}$, $\{5,21\}$, $\{5,29\}$, $\{5,30\}$, $\{5,35\}$, $\{5,37\}$, $\{5,42\}$, $\{5,125\}$, $\{6,15\}$, $\{6,54\}$, $\{7,10\}$, $\{7,15\}$, $\{7,23\}$, $\{7,28\}$, $\{7,31\}$, $\{7,39\}$, $\{7,55\}$, $\{10,15\}$, $\{10,26\}$, $\{10,40\}$, $\{10,58\}$, $\{10,250\}$, $\{14,30\}$, $\{14,56\}$, $\{14,78\}$.

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- Bhargava's escalator method is used to reduce problems like those above to a finite calculation involving specific quaternary quadratic forms.

- The modular symbols algorithm can be used to decompose $C(z)$ into newforms and to derive an explicit bound on $a_C(n)$.

- With that in hand, one can determine the integers represented by a form Q.

Example (from 451 theorem)

- For

$$Q(x, y, z, w) = x^2 - xy + 2y^2 + yz - 2yw + 5z^2 + zw + 29w^2$$

we have $\theta_Q \in M_2(\Gamma_0(4200), \chi_{168})$.

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- Once this is known, it takes 10 seconds to check that Q represents every odd number.

Another method (1/5)

• When $D(Q)$ is a fundamental discriminant, there's another method. Rather than explicitly compute the decomposition of the cusp form part into newforms, we instead do the following.

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• Compute an upper bound for $\langle C, C \rangle$. We do this via the form $Q^{*} = \frac{1}{2}\vec{x}^{T} N(Q) A^{-1} \vec{x}$. Let $\theta_{Q^{*}} = E^{*} + C^{*}$.

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- Compute a lower bound on $\langle g, g \rangle$ for all newforms $g \in S_2(\Gamma_0(D), \chi_{D(Q)})$.

Another method (2/5)

Theorem

If g is a non-CM newform in $S_2(\Gamma_0(D(Q)), \chi_{D(Q)})$, then

$$\langle g, g \rangle \geq \frac{1}{685 \log(D(Q))} \left(\prod_{p \mid D(Q)} \frac{p}{p+1} \right)$$

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- There might be CM newforms in $S_2(\Gamma_0(D(Q)), \chi_{D(Q)})$. We explicitly enumerate them and handle them separately.

Another method (3/5)

- Define

$$\psi(x) = -\frac{6}{\pi} x K_1(4\pi x) + 24x^2 K_0(4\pi x).$$

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Theorem

If $C(z)$ is the cusp form part of θ_Q, then $\langle C, C \rangle$ is given by

$$\sum_{n=1}^{\infty} \frac{2^{\omega(\gcd(n, D(Q)))} N(Q) a_{C^*}(n)^2}{n[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(D(Q))]} \sum_{d=1}^{\infty} \psi\left(d\sqrt{\frac{n}{D(Q)}}\right).$$

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$$Q(x, y, z, w) = x^2 + 3y^2 + 3yz + 3yw + 5z^2 + zw + 34w^2$$

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- We find that for all newforms g,

$$\langle g, g \rangle \geq 1.019 \cdot 10^{-5}.$$

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- We compute the first 101700 coefficients of C^* and use the formula from three slides back to get

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• Checking up to this bound requires 22 minutes and 29 seconds. We find that Q represents all odd numbers.

Summary

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- Stronger bounds can be obtained if $D(Q)$ is a fundamental discriminant.

- These methods can be used to determine precisely which integers are represented by quaternary quadratic forms of large level.

That's all

Thank you very much!