Statistics for Partitions and Unimodal Sequences

Walter Bridges (LSU)

September 17, 2020
Vanderbilt Number Theory Seminar

This talk is based on:

- W. Bridges "Distinct parts partitions with bounded largest part," *to appear in Research in Number Theory* arXiv: 2004.12036
- W. Bridges "Limit shapes for unimodal sequences," arXiv: 2001.06878

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Goals:

1. discuss **probabilistic methods** for proving asymptotics

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1. discuss **probabilistic methods** for proving asymptotics
2. describe "direct" generating functions approach to limit shapes for unimodal sequences

Introduction

Partitions

Definition

A *partition* of n is a multiset of positive integers $\{\lambda_k\}$ with

$$\lambda_1 \geq \cdots \geq \lambda_\ell \qquad \text{and} \qquad \sum_{k=1}^{\ell} \lambda_k = n.$$

We write $\lambda \vdash n$ and denote the *size* as $|\lambda| := n$. We set $p(n) := \#\{\lambda \vdash n\}$.

Example

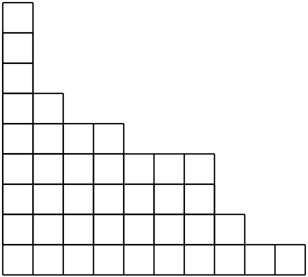
The partitions of 4 are

$$4, \;\; 3+1, \;\; 2+2, \;\; 2+1+1, \;\; 1+1+1+1,$$

so $p(4) = 5$.

Young/Ferrer's Diagrams

Represent parts as columns of squares.



$$9 + 6 + 5 + 5 + 4 + 4 + 4 + 2 + 1 + 1 = 41.$$

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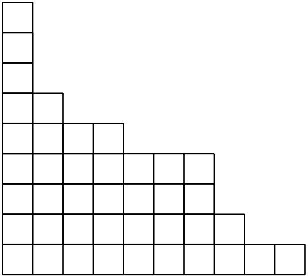

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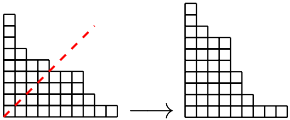
Conjugation:



Asymptotic enumeration

Question

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$$p(n) = \text{Coeff}[q^n]\left(\prod_{k\geq 1}\frac{1}{1-q^k}\right). \quad (= \frac{q^{1/24}}{\eta(\tau)}, \text{ modular})$$

Asymptotic enumeration

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Theorem (Hardy-Ramanujan 1919, Rademacher 1937)

$p(n) =$ *"convergent series"; in particular,*

$$p(n) \sim \frac{1}{4\sqrt{3}n}e^{\pi\sqrt{\frac{2n}{3}}}.$$

Remark

Hardy-Ramanujan's work marks the birth of the HR-Circle Method.

Other asymptotic formulas

$d(n)$ - "distinct parts partitions of n"

$= \text{Coeff}[q^n] \prod_{k\geq 1}(1+q^k)$. (modular)

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Other asymptotic formulas

$p_t(n)$ - partitions with largest part $\leq t\sqrt{n}$

$$= \text{Coeff}[q^n] \prod_{k \leq t\sqrt{n}} \frac{1}{1-q^k}. \quad \text{(non-modular)}$$

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Theorem (Szekeres 1953, Canfield 1997, Romik 2005)

Let $\alpha = \alpha(t)$ satisfy $\int_0^t \frac{ue^{-\alpha u}}{1-e^{-\alpha u}} du = 1$. Then we have an asymptotic of the form

$$p_t(n) \sim \frac{g(\alpha, t)}{n} e^{H(\alpha,t)\sqrt{n}}.$$

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- Canfield: recurrences + real analysis
- Romik: probabilistic

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- Szekeres: **saddle-point method** (complex analysis)
- Canfield: recurrences + real analysis
- Romik: **probabilistic**

(More intuitive reformulation of saddle-point method.)

$d_t(n)$ - distinct parts partitions with largest part $\leq t\sqrt{n}$
$= \mathrm{Coeff}[q^n] \prod_{k \leq t\sqrt{n}} (1+q^k)$.

Theorem (B. (2020))

Let $t > \sqrt{2}$ and let $\beta = \beta(t)$ satisfy $\int_0^t \frac{ue^{-\beta u}}{1+e^{-\beta u}} du = 1$. Then

$$d_t(n) \sim \frac{A_n(t)}{n^{3/4}} e^{B(t)\sqrt{n}},$$

where

$$A_n(t) = \frac{e^{\frac{\beta t}{2}} + e^{-\frac{\beta t}{2}}}{2(1+e^{-\beta t})^{\{t\sqrt{n}\}}} \sqrt{\frac{\beta'(t)}{\pi t}}, \qquad B(t) = 2\beta + \log(1+e^{-\beta t}).$$

- **probabilistic** proof

Largest part distribution

$P_n(\lambda) := \frac{1}{p(n)}$ - uniform probability meas. on $\{\lambda \vdash n\}$.

Question

What is the size of λ_1 for a typical partition of n?

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Theorem (Erdős-Lehner 1941)

Let $c := \frac{\sqrt{6}}{\pi}$. Then

$$\lim_{n\to\infty} P_n\left(\frac{\lambda_1 - c\sqrt{n}\log\left(c\sqrt{n}\right)}{c\sqrt{n}} \leq x\right) = e^{-e^{-x}}.$$

Answer

Typically, $\lambda_1 \sim c\sqrt{n}\log(c\sqrt{n})$ and the error has an extreme value distribution.

Largest part distribution

Proof Idea

By inclusion-exclusion:

$$\begin{aligned}\#\{\lambda \vdash n : \lambda_1 \leq k\} = p(n) &- \sum_{r_1 \geq 1} p\left(n - (k + r_1)\right) \\ &+ \sum_{r_2 > r_1 \geq 1} p\left(n - (k + r_1) - (k + r_2)\right) \\ &- \ldots .\end{aligned}$$

Set $k := c\sqrt{n} \log\left(c\sqrt{n}\right) + c\sqrt{n}x$ *and plug in Hardy-Ramanujan asymptotic.*

t-th largest part distribution

Theorem (Fristedt 1993, Trans. AMS)

$$\lim_{n\to\infty} P_n\left(\frac{\lambda_t - c\sqrt{n}\log\left(c\sqrt{n}\right)}{c\sqrt{n}} \leq x\right) = \frac{1}{(t-1)!}\int_{-\infty}^{x} e^{-e^{-u}-tu}du.$$

Fristedt also found joint distribution for $(\lambda_1, \ldots, \lambda_{t_n})$ where $t_n = o\left(n^{1/4}\right)$ and distributions for **many** other statistics.

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Proof Idea

Fristedt's **conditioning device**. Let N, the size of a partition, be a *random variable*. Then

$$P_n(\cdot) = Q_q(\cdot|N = n),$$

for a "better" family of probability measures Q_q.

Limit shapes

Question

What are the likely shapes of diagrams among partitions of n?

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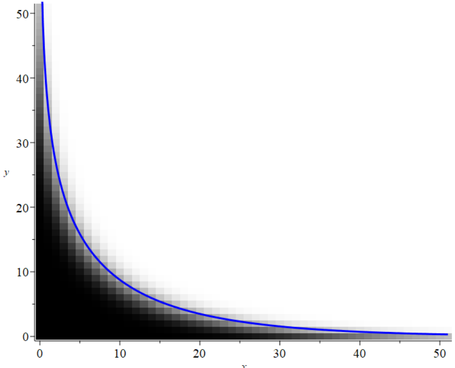
Figure: Density plot of $\{\lambda \vdash 300\}$.



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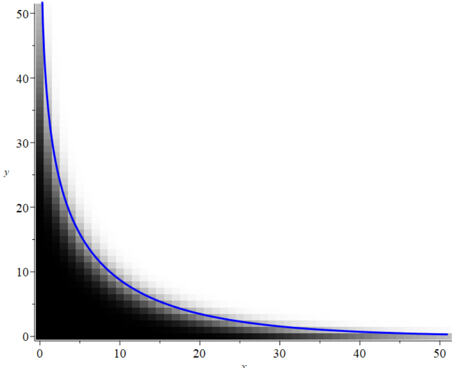
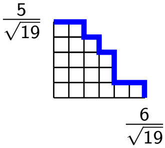


$\widetilde{\varphi}(\lambda)$ - renormalized shape; rescale by $\frac{1}{\sqrt{n}}$



Total Area $= 1$

Limit shapes

Question

What are the likely shapes of diagrams among partitions of n?

Let $\epsilon > 0$ and let N_ϵ be an ϵ-n'hood of $e^{-\frac{\pi}{\sqrt{6}}x} + e^{-\frac{\pi}{\sqrt{6}}y} = 1$.

Conjecture (Temperley 1952, Szalay-Turán 1977, Vershik 1996, ...)

(Roughly)

$$\lim_{n\to\infty} P_n\left(\widetilde{\varphi}(\lambda) \subset N_\epsilon\right) = 1.$$

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- Dembo-Vershik-Zeitouni 1998 proof of above* and large deviation principle using Fristedt's conditioning device
- Petrov 2010 proof by "elementary" gen. f'n estimates
- distinct parts partitions: $e^{-\frac{\pi}{\sqrt{6}}y} - e^{-\frac{\pi}{\sqrt{6}}x} = 1$

Question

Are there limit shapes for unimodal sequences?

Unimodal Sequences

Definition

A *unimodal sequence* λ of size n is a sequence of positive integers satisfying

$$\lambda_1 \leq \cdots \leq \lambda_p \geq \cdots \geq \lambda_\ell \qquad \sum_{k=1}^{\ell} \lambda_k = n.$$

λ_p (and other parts of this size) is called a *peak*.

Example

The unimodal sequences of size 4 are

$$1+1+1+1, 2+1+1, 1+2+1, 1+1+2, 2+2, 3+1, 1+3, 4,$$

so $u(4) = 8$.

Because of peaks, unimodal sequences are **not** “double partitions”,

$$\sum_{n\geq 1} u(n)q^n = \underbrace{\prod_{k\geq 1}\frac{1}{(1-q^k)^2}}_{\text{modular}} \underbrace{\sum_{k\geq 1}(-1)^{k+1}q^{\frac{k(k+1)}{2}}}_{\text{false } \theta\text{-f'n}} .$$

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Theorem (Auluck 1951)

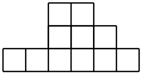
$u(n) \sim \frac{1}{2^3 3^{3/4} n^{5/4}} e^{2\pi\sqrt{\frac{n}{3}}}$

Theorem (Bringmann-Nazaroglu 2019, Research in Math. Sci.)

$u(n) =$ *"convergent series"*

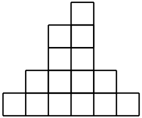
Definitions (by Picture)

- (**Unrestricted**) U.S.



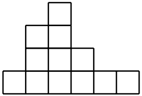
$(1, 1, 3, 3, 2, 1)$

- **Strongly** U.S.
  - *strict ineq.*



$(1, 2, 4, 5, 2, 1)$

- **Semi-strict** U.S.
  - *a single peak; strict ineq. to the left*



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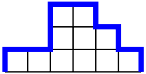

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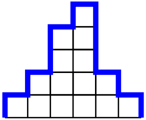

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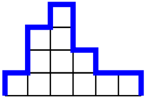

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$\widetilde{\varphi}(\lambda)$ - renormalized shape, rescale by $\frac{1}{\sqrt{n}}$

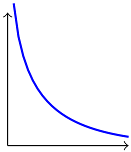
Limit shapes

Question

What is the typical shape of a unimodal sequence of size n?

Recall limit shape for partitions:

$$e^{-\frac{\pi}{\sqrt{6}}x} + e^{-\frac{\pi}{\sqrt{6}}y} = 1.$$



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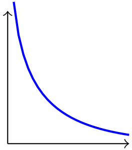
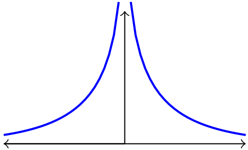
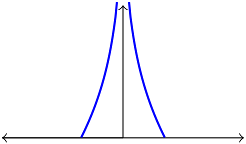


- So we may expect:



unrestricted u. s.



strongly u.s.

?

semi-strict u.s.

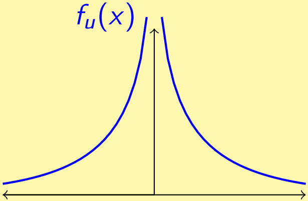
Limit shapes

Theorem (B.)

We have the following limit shapes:

- *Unrestricted Unimodal Sequences*

$$f_u(x) := \begin{cases} -\frac{\sqrt{3}}{\pi}\log\left(1-e^{\frac{\pi}{\sqrt{3}}x}\right) & \text{if } x<0, \\ -\frac{\sqrt{3}}{\pi}\log\left(1-e^{-\frac{\pi}{\sqrt{3}}x}\right) & \text{if } x>0. \end{cases}$$



- Each side is the limit shape for partitions (scaled by $\frac{1}{2}$).

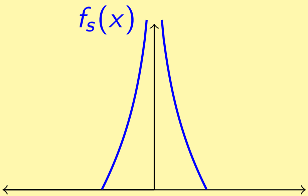
Limit shapes

Theorem (B.)

(continued)

- *Strongly Unimodal Sequences*

$$f_s(x) := \begin{cases} -\frac{\sqrt{6}}{\pi}\log\left(e^{-\frac{\pi}{\sqrt{6}}x}-1\right) & \text{if } x \in \left[-\frac{\sqrt{6}}{\pi}\log(2), 0\right), \\ -\frac{\sqrt{6}}{\pi}\log\left(e^{\frac{\pi}{\sqrt{6}}x}-1\right) & \text{if } x \in \left(0, \frac{\sqrt{6}}{\pi}\log(2)\right]. \end{cases}$$



- Each side - lim shape for dist. parts partitions (scaled by $\frac{1}{2}$).

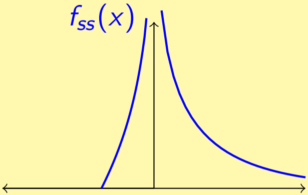
Limit shapes

Theorem (B.)

(continued)

- *Semi-strict Unimodal Sequences*

$$f_{ss}(x) := \begin{cases} -\frac{2}{\pi}\log\left(e^{-\frac{\pi}{2}x} - 1\right) & \text{if } x \in \left[-\frac{2}{\pi}\log(2), 0\right), \\ -\frac{2}{\pi}\log\left(1 - e^{-\frac{\pi}{2}x}\right) & \text{if } x > 0. \end{cases}$$



Remark: Left Area + Right Area $= \frac{1}{3} + \frac{2}{3} = 1$.

Interpretation

- 0-1 laws for "medium" parts ($\asymp \sqrt{n}$)

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- E.g., fix $\epsilon > 0$. Then for 100% of strongly unimodal sequences as $n \to \infty$, the number of parts lies in

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$$\sqrt{n}\left[2\frac{\sqrt{6}}{\pi}\log(2)-\epsilon, 2\frac{\sqrt{6}}{\pi}\log(2)+\epsilon\right].$$

- Limit shapes say very little about "small" and "large" parts.

Consequences: semi-strict ranks

$\mathcal{SS}(n)$ - semi-strict unimodal sequences of size n

$\text{rank}(\lambda) = \#$ right parts $- \#$ left parts (symmetry)

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Theorem (Bringmann–Jennings-Shaffer–Mahlburg, (2020))

The rank of semi-strict unimodal sequences has a point-mass distribution:

$$\lim_{n\to\infty} \frac{1}{\#\mathcal{SS}(n)} \# \left\{ \lambda \in \mathcal{SS}(n) : \frac{rank(\lambda)}{\frac{\sqrt{n}\log(n)}{\pi}} \leq x \right\} = \begin{cases} 0 & \text{if } x < 1, \\ 1 & \text{if } x \geq 1. \end{cases}$$

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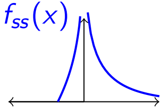
Recall limit shape:



Area : $\frac{1}{3} + \frac{2}{3} = 1$

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$\frac{1}{3} + \frac{2}{3}$ heuristic:

$$rank(\lambda) \approx \# \left(\text{parts in partition of } \frac{2}{3} n \right) - \frac{2}{\pi} \log 2\sqrt{n}$$

$$\sim \underbrace{\frac{\sqrt{3}}{\sqrt{2}\pi} \log\left(\frac{2}{3}n\right)\sqrt{\frac{2}{3}n}}_{\text{Erdős-Lehner}} - \frac{2}{\pi}\log 2\sqrt{n} \sim \frac{\sqrt{n}\log n}{\pi}.$$

Consequences: overpartitions

Question

Are there limit shapes for overpartitions?

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Definition (Corteel-Lovejoy 2004)

An *overpartition* of n is a partition in which the last occurrence of a part may (or may not) be over-marked.

Example

The overpartitions of 4 are

$$4, \; \overline{4}, \; 3+1, \; \overline{3}+1, \; \overline{3}+\overline{1}, \; 2+2, \; 2+\overline{2}, \; 2+1+1,$$

$$\overline{2}+1+1, \; \overline{2}+1+\overline{1}, \; 1+1+1+1, \; 1+1+1+\overline{1},$$

so $\overline{p}(4) = 12$.

Consequences: overpartitions

- DeSalvo-Pak 2019 - geometrically nice bijections $\implies$ transfer of limit shapes

Consequences: overpartitions

- DeSalvo-Pak 2019 - geometrically nice bijections $\Longrightarrow$ transfer of limit shapes
- $\overline{p}(n) = ss(n) + ss(n+1)$. Bijective proof:

Consequences: overpartitions

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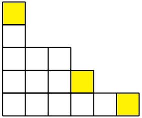
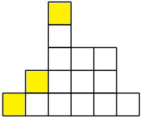
Case 1: a single marked largest part



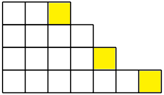
$\in \overline{\mathcal{P}}(n)$

$\longrightarrow$



$\in \mathcal{SS}(n)$

Case 2: at least 1 unmarked largest part



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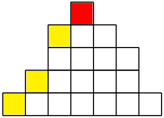
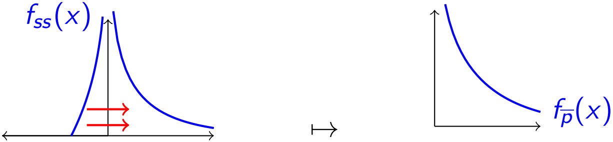


$\in \mathcal{SS}(n+1)$

Limit shape for overpartitions

Thus, we can transfer the limit shapes:



Corollary (B.)

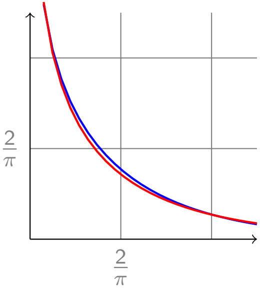
The limit shape for overpartitions is

$$f_{\overline{p}}(x) := \frac{2}{\pi} \log \left(\frac{1 + e^{-\frac{\pi}{2}x}}{1 - e^{-\frac{\pi}{2}x}} \right).$$

- symmetric in x and y (due to conjugation)

Limit shape for overpartitions

Comparison of limit shapes for partitions and overpartitions:



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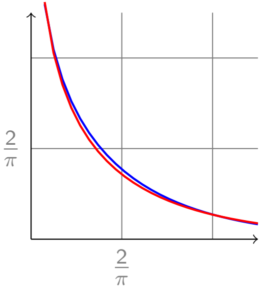

Question

- *Is there a more direct combinatorial explanation for this difference?*

Limit shape for overpartitions

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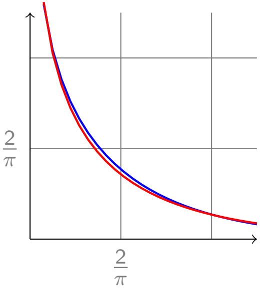

Question

- *Is there a more direct combinatorial explanation for this difference?*
- *How should we alter definition of partitions to achieve certain effects in limit shape?*

Proof of asymptotic for $d_t(n)$

Probabilistic proof of $d_t(n)$ asymptotic

Goal: asymptotic formula for

$$d_t(n) = \text{Coeff}[q^n]\mathcal{D}_t(q), \qquad \text{where } \mathcal{D}_t(q) := \prod_{k \le t\sqrt{n}} (1 + q^k).$$

The measures Q_q

- For $q \in (0,1)$, set

$$Q_q(\lambda) := q^{|\lambda|} \prod_{k \leq t\sqrt{n}} \frac{1}{1+q^k} = q^{|\lambda|} \mathcal{D}_t(q)^{-1}.$$

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- r.v. $N := \sum_{k \ge 1} kX_k$ - the *size*.
  - (e.g. $N(6+3+1) = 10$.)

The measures Q_q

Facts

1. X_k's are **independent** under Q_q and

$$Q_q(N = n) = d_t(n) q^n \mathcal{D}_t(q)^{-1}.$$

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$$\text{saddle-point equation} \quad \longleftrightarrow \quad E_q(N) \sim n.$$

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- Let $\sigma_n :=$ std. dev. of N. Then for $q = e^{-\beta/\sqrt{n}}$,

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Heuristic:

$$\begin{aligned} Q_q(N=n) &= Q_q\left(-\frac{1}{2\sigma_n} \leq \frac{N-n}{\sigma_n} \leq \frac{1}{2\sigma_n}\right) \\ &\approx \frac{1}{\sqrt{2\pi}} \int_{-\frac{1}{2\sigma_n}}^{\frac{1}{2\sigma_n}} e^{-x^2/2} dx \\ &\sim \frac{1}{\sqrt{2\pi}\sigma_n}. \end{aligned}$$

With $q = e^{-\beta/\sqrt{n}}$ so that $E_q(N) \sim n$,

$$d_t(n) = Q_q(N = n) q^{-n} \mathcal{D}_t(n),$$

we use

Euler-Maclaurin summation
$+$
Fourier inversion of char. f'n $\widetilde{\varphi}_N$.

$$d_t(n) \sim \frac{A_n(t)}{n^{3/4}} e^{B(t)\sqrt{n}}.$$

Boltzmann models

Duchon-Flajolet-Louchard-Schaeffer 2003: “Boltzmann models” for general combinatorial structures.

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Duchon-Flajolet-Louchard-Schaeffer 2003: "Boltzmann models" for general combinatorial structures. If

- $\mathscr{C}_n := \{\gamma : |\gamma| = n\}$
- $\mathscr{C} = \cup_{n \geq 1} \mathscr{C}_n$,

study uniform measure $P_n(\gamma) := \frac{1}{\#\mathscr{C}_n}$ using the **Boltzmann model**,

$$Q_q(\gamma) := \frac{q^{|\gamma|}}{\sum_{\omega \in \mathscr{C}} q^{|\omega|}}.$$

- Allowing *size* to be a random variable leads to faster sampling algorithms.

Boltzmann models

- Gen. f'n for unimodal sequences is **not** an infinite product.
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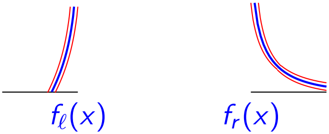
Boltzmann models

- Gen. f'n for unimodal sequences is **not** an infinite product.
- Boltzmann model is less useful*.
- Need "direct" generating functions approach for statistics for unimodal sequences.

Proof of limit shapes for semi-strict unimodal sequences

Proof Sketch for Semi-strict Unimodal Sequences

- Step 1: Limit shapes for the **left and right halves** in isolation: as $n \to \infty$, a proportion of **0% are not** in $N_\epsilon(f_\ell)$ and $N_\epsilon(f_r)$.



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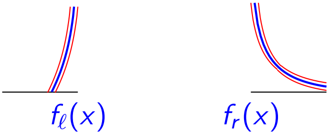

- Step 2: To avoid degenerate limit shape, show **peaks** are typically $\omega(\sqrt{n})$.

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For a **unique** κ with $q = e^{-\kappa/\sqrt{n}}$,

$$\frac{q^{-n}\mathcal{SS}(q)}{ss(n)} = e^{o(\sqrt{n})}.$$

Now choose $z \neq 1$ so that $V(z) = e^{-C_0\sqrt{n}}$ **uniformly** for all pairs (a, b) with $\lambda\left(\frac{1}{\sqrt{n}}(a, b)\right) \notin N_\epsilon(f)$.

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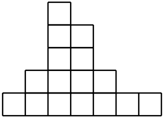
$$n^2 e^{-C_0\sqrt{n}+o(\sqrt{n})} \to 0.$$

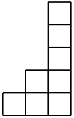
Step 2: Peaks are $\omega(\sqrt{n})$

- Show $\frac{ss_t(n)}{ss(n)} \to 0$ where $ss_t(n)$ counts sequences with **peak** $\leq t\sqrt{n}$ by **injecting** into **pairs of partitions**:

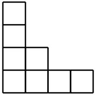


$\mathcal{SS}_t(n)$

$\hookrightarrow$



$\mathcal{D}_t(n)$



$\mathcal{P}_t(n)$

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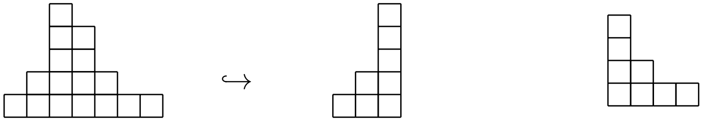
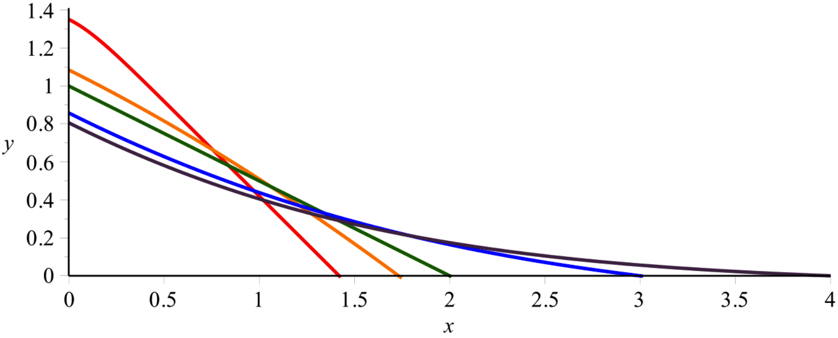


$\mathcal{SS}_t(n) \hookrightarrow \mathcal{D}_t(n) \quad \mathcal{P}_t(n)$

- Apply asymptotics of Szekeres and B., respectfully.

Essential ingredients for limit shapes for u.s.

- Comb'lly, analytically "nice" gen. f'n $A(q) = \sum_{n \geq 1} a(n) q^n$.
- Choice of q such that $\frac{q^{-n} A(q)}{a(n)} = e^{o(\log a(n))}$.
- Ability to "glue together" shapes by showing peaks are $\omega(\sqrt{n})$ (or similar).



- Apparent limit shapes for $\mathcal{D}_t(n)$ for $t = \sqrt{2}, \sqrt{3}, 2, 3, 4$.
- Concavity switches at $t = 2$. Explanation?

Thanks for listening!