Multiplicative theory of (additive) partitions

Robert Schneider
University of Georgia

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Additive number theory

Patterns and interconnections

Patterns and interconnections

- theory of *partitions*

Patterns and interconnections

- theory of *partitions*
- beautiful generating functions
- surprising bijections
- Ramanujan congruences
- combinatorics, algebra, analytic num. theory, mod. forms, stat. phys., QT, string theory, chemistry, ...

Birth of partition theory

Birth of partition theory

Ishango bone (Africa, ca. 20,000 B.C.E.)

Birth of partition theory

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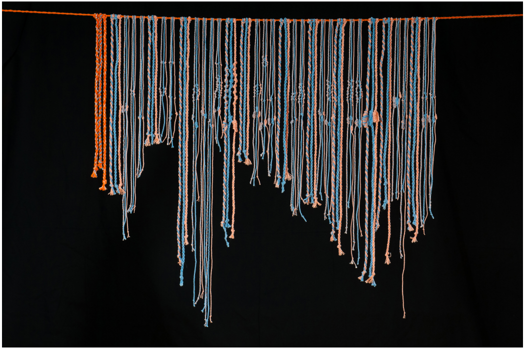
Ahmes papyrus (Egypt, ca. 2,000 B.C.E.)

Partition theory

Partition theory



Incan *quipu* (Peru, 2,000 B.C.E. - 1600s)

Birth of partition theory

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G. W. Leibniz (1600s)

Birth of partition theory

Leibniz

- wondered about size of $p(n) := \#$ of partitions of n

Birth of partition theory

Leibniz

- wondered about size of $p(n) :=$ # of partitions of n
- $p(n)$ is called the *partition function*

Birth of partition theory

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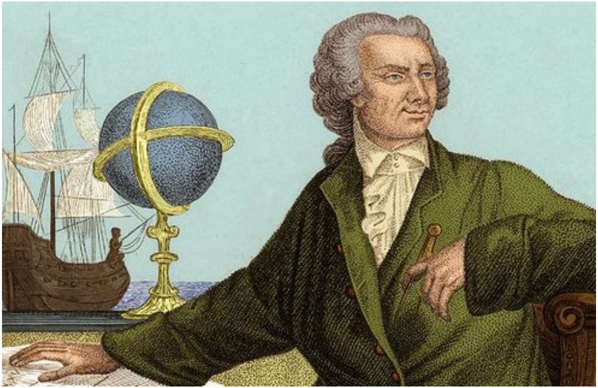

Leonhard Euler (1700s)

Partition generating function (Euler)

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$$\sum_{n=0}^{\infty} p(n)q^n$$

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$$\sum_{n=0}^{\infty} p(n) q^n = \prod_{n=1}^{\infty} (1 - q^n)^{-1} \qquad (q \in \mathbb{C}, |q| < 1)$$

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Template for partition theory

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- product-sum generating functions

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- product-sum generating functions
- combinatorics encoded in exponents, coefficients

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- product-sum generating functions
- combinatorics encoded in exponents, coefficients
- connected analysis to partitions

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RHS

Birth of partition theory

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RHS = $\prod_{n=1}^{\infty}(1 + q^n + q^{2n} + q^{3n} + \dots)$

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RHS = $\prod_{n=1}^{\infty}(1+q^n+q^{2n}+q^{3n}+\ldots)$ (geom. series)
= $\prod_{n=1}^{\infty}(1+q^n+q^{n+n}+q^{n+n+n}+\ldots)$

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= $1 + q^1 + q^{1+1} + q^2 + q^{1+1+1} + q^{1+2} + q^3 + \ldots$= LHS

Multiplicative number theory

I.e., most of classical number theory

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- primes
- divisors
- Euler phi function $\varphi(n)$, Möbius function $\mu(n)$
- arithmetic functions, Dirichlet convolution
- zeta functions, Dirichlet series, L-functions

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Eratosthenes, Euclid (Alexandria, ca. 300 B.C.E.)

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- explicit zeta values $\rightarrow$ compute even powers of π:

$$\zeta(2) = \frac{\pi^2}{6}, \quad \zeta(4) = \frac{\pi^4}{90}, \quad \zeta(6) = \frac{\pi^6}{945}, \ldots$$

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- "Euler product" generating functions
- connected analysis to prime numbers

Additive-multiplicative correspondence (Euler)

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Partition generating function (encodes addition)

$$\prod_{n=1}^{\infty}(1-q^n)^{-1} = \sum_{n=0}^{\infty} p(n)q^n, \quad |q|<1$$

Euler product formula (encodes primes / multiplic.)

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- Proofs feel similar (multiply geometric series)

Alladi–Erdős (1970s)

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Bijection between *integer factorizations*, *prime partitions*

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Are other thms. in arithmetic *images in prime partitions* of combinatorial/set-theoretic meta-structures?

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Is there an *algebra of partitions* generalizing arithmetic in integers

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- inspired by lattice theory
- unifies classical results on gen. functions, bijections
- suggestive of a *universal algebra of partitions*

Question

Is there an *algebra of partitions* generalizing arithmetic in integers (i.e., prime partitions)?

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- Expect partition properties $\rightarrow$ properties of integers

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- Let $\mathcal{P}_{\mathbb{X}}$ denote partitions into elements $\lambda_i \in \mathbb{X} \subseteq \mathbb{N}$, e.g. $\mathcal{P}_{\mathbb{P}}$ is the "prime partitions".

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- Define $\ell(\emptyset) = |\emptyset| = m_i(\emptyset) = 0, \quad \emptyset \vdash 0$.

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- See “The product of parts or ‘norm’ etc.” (S-Sills)

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Partition multiplication

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- Identity is $\emptyset$
- Partitions into one part are like primes, FTA holds

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- Replace $\mathcal{P}$ with $\mathcal{P}_{\mathbb{P}} \to$ mult./div. in $\mathbb{Z}^+$

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- Replacing $\mathcal{P}$ with $\mathcal{P}_{\mathbb{P}}$ reduces to $\mu(N(\lambda))$, where $N(\lambda)$ is the norm (product of parts).

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$$\sum_{\delta|\lambda} \mu_{\mathcal{P}}(\delta) = \begin{cases} 1 & \text{if } \lambda = \emptyset, \\ 0 & \text{otherwise} \end{cases}$$

Parallel universe

Partition Möbius inversion

If we have

$$f(\lambda) = \sum_{\delta|\lambda} g(\delta)$$

we also have

$$g(\lambda) = \sum_{\delta|\lambda} \mu_{\mathcal{P}}(\lambda/\delta) f(\delta).$$

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$$\sum_{\delta|\lambda} \varphi_{\mathcal{P}}(\delta) = N(\lambda), \qquad \varphi_{\mathcal{P}}(\lambda) = N(\lambda) \sum_{\delta|\lambda} \frac{\mu_{\mathcal{P}}(\delta)}{N(\delta)}$$

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- Replacing $\mathcal{P}$ with $\mathcal{P}_{\mathbb{P}}$ reduces to classical cases.
- Many analogs of objects in multiplic. # theory...

Partition Cauchy product

$$\left(\sum_{\lambda\in\mathcal{P}} f(\lambda)q^{|\lambda|}\right)\left(\sum_{\lambda\in\mathcal{P}} g(\lambda)q^{|\lambda|}\right) = \sum_{\lambda\in\mathcal{P}} q^{|\lambda|}\sum_{\delta|\lambda} f(\delta)g(\lambda/\delta)$$

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Other multiplicative objects generalize to partition theory...

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- For $1 \notin \mathcal{P}' = \mathcal{P}_{\mathbb{X}}$ (parts in $\mathbb{X} \subset \mathbb{N}$)

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For $\mathcal{P}' \subsetneq \mathcal{P}, s \in \mathbb{C}$, define a *partition zeta function*:

$$\zeta_{\mathcal{P}'}(s) := \sum_{\lambda \in \mathcal{P}'} N(\lambda)^{-s}$$

- $N(\lambda)$ is the *norm* (product of parts) of λ, $N(\emptyset) := 1$.
- For $1 \notin \mathcal{P}' = \mathcal{P}_{\mathbb{X}}$ (parts in $\mathbb{X} \subset \mathbb{N}$) $\rightarrow$ Euler product:

$$\zeta_{\mathcal{P}_{\mathbb{X}}}(s) = \prod_{n \in \mathbb{X}} (1 - n^{-s})^{-1}$$

Fix $s = 2$, vary subset $\mathcal{P}'$

Partition zeta functions: nice identities

Fix $s = 2$, vary subset $\mathcal{P}'$

Summing over partitions into prime parts:

$$\zeta_{\mathcal{P}_{\mathbb{P}}}(2) = \zeta(2) = \frac{\pi^2}{6}$$

Summing over partitions into even parts:

$$\zeta_{\mathcal{P}_{\text{even}}}(2) = \frac{\pi}{2}$$

Summing over partitions into distinct parts:

$$\zeta_{\mathcal{P}_{\text{distinct}}}(2) = \frac{\sinh \pi}{\pi}$$

Partition zeta functions: nice identites

Partition analogs of classical identities

$$\sum_{\lambda \in \mathcal{P}_{\mathbb{X}}} \mu_{\mathcal{P}}(\lambda) N(\lambda)^{-s} = \frac{1}{\zeta_{\mathcal{P}_{\mathbb{X}}}(s)}$$

$$\sum_{\lambda \in \mathcal{P}_{\mathbb{X}}} \varphi_{\mathcal{P}}(\lambda) N(\lambda)^{-s} = \frac{\zeta_{\mathcal{P}_{\mathbb{X}}}(s-1)}{\zeta_{\mathcal{P}_{\mathbb{X}}}(s)}$$

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- Different subsets of $\mathcal{P}$ induce very diff. zeta values

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Takeaway from these examples

- Different subsets of $\mathcal{P}$ induce very diff. zeta values
- Classical zeta theorems $\rightarrow$ partition zeta theorems

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Do there exist (non-Riemann) partition zeta functions such that, for the "right" choice of $\mathcal{P}' \subsetneq \mathcal{P}$,

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$$\zeta_{\mathcal{P}}(\{s\}^k) := \sum_{\substack{\lambda \in \mathcal{P} \\ \ell(\lambda)=k}} N(\lambda)^{-s} \qquad (\mathrm{Re}(s) > 1)$$

Partition zeta functions: nice family

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Summing over partitions of fixed length $k > 0$:

$$\zeta_{\mathcal{P}}(\{2\}^k) = \frac{2^{2k-1}-1}{2^{2k-2}}\zeta(2k) = \pi^{2k} \times \textit{rational number}$$

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Theorem (Ono-Rolen-S, 2017)

For $N \geq 1$: $\zeta_{\mathcal{P}}(\{2N\}^k) = \pi^{2Nk} \times \textit{rational number}$

Partition zeta functions: nice family

Theorem (Ono-Rolen-S., 2017)

Some *other* partition zeta fctns. contin. to right half-plane.

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Natural questions

Partition zeta functions: nice family

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Some *other* partition zeta fctns. contin. to right half-plane.

Natural questions

General $\zeta_{\mathcal{P}}(\{s\}^k)$? Analytic continuation? Poles? Roots?

Theorem (S.-Sills, 2020)

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For $\mathrm{Re}(s) > 1$:

$$\zeta_{\mathcal{P}}(\{s\}^k) = \sum_{\lambda \vdash k} \frac{\zeta(s)^{m_1}\zeta(2s)^{m_2}\zeta(3s)^{m_3}\cdots\zeta(ks)^{m_k}}{N(\lambda)\; m_1!\; m_2!\; m_3!\; \cdots\; m_k!}$$

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Partition zeta functions: analytic properties

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Note: $\zeta_{\mathcal{P}}(\{s\}^k)$ **not** zero at roots of $\zeta(s)$ for $k > 1$

MacMahon-partition zeta correspondence

Proof mimics Sills’ combinatorial proof (2019) of *MacMahon’s partial fraction decomposition*...

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MacMahon's partial fraction decomposition

For $|q| < 1$:

$$\prod_{n=1}^{k}(1-q^n)^{-1} = \sum_{\lambda \vdash k} \frac{(1-q^2)^{-m_2} \cdots q^{km_k}(1-q^k)^{-m_k}}{N(\lambda)\ m_1!\ m_2!\ \cdots\ m_k!}$$

Although here we really want to use...

MacMahon's partial fraction decomposition times q^k

$$q^k \prod_{n=1}^{k} (1-q^n)^{-1} = \sum_{\ell(\lambda)=k} q^{|\lambda|}$$

$$= \sum_{\lambda \vdash k} \frac{q^{m_1}(1-q)^{-m_1} \cdot q^{2m_2}(1-q^2)^{-m_2} \cdots q^{km_k}(1-q^k)^{-m_k}}{N(\lambda)\, m_1!\, m_2! \, \cdots \, m_k!}$$

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- RHS = comb. geom. series over partitions of *size* k

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Compare and contrast

$$\zeta_{\mathcal{P}}(\{s\}^k) = \sum_{\ell(\lambda)=k} N(\lambda)^{-s} = \sum_{\lambda \vdash k} \frac{\zeta(s)^{m_1}\zeta(2s)^{m_2}\cdots\zeta(ks)^{m_k}}{N(\lambda)\ m_1!\ m_2!\ \cdots\ m_k!}$$

MacMahon-partition zeta correspondence

Gen fctn component	Analogous zeta fctn component
$q^{\lvert\lambda\rvert}$	$N(\lambda)^{-s}$
$\frac{q^j}{1-q^j}$	$\zeta(js)$
$\frac{q^k}{\prod_{j=1}^{k}(1-q^j)}$	$\zeta_{\mathcal{P}}(\{s\}^k)$

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Multiplication of terms of either shape q^n or n^{-s} generates partitions in exactly the same way:

$$q^{\lambda_1}q^{\lambda_2}q^{\lambda_3}\cdots q^{\lambda_r} = q^{|\lambda|} \quad \longleftrightarrow \quad \lambda_1^{-s}\lambda_2^{-s}\lambda_3^{-s}\cdots\lambda_r^{-s} = N(\lambda)^{-s}$$

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$\frac{q^j}{1-q^j}$	$\zeta(js)$
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The term q^{jn} in geom series $\sum_{n=1}^{\infty} q^{jn}$ and resp. term n^{-js} of $\zeta(js)$ both encode partition $(n)^j := (n, n, ..., n)$ (j times):

$$\frac{q^j}{1-q^j} = \sum_{n=1}^{\infty} q^{\lvert (n)^j \rvert} \quad \longleftrightarrow \quad \zeta(js) = \sum_{n=1}^{\infty} N\left((n)^j\right)^{-s}.$$

Geometric series-zeta function duality

MacMahon-partition zeta correspondence

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Correspondence says $\frac{q^j}{1-q^j}$ and $\zeta(js)$ **are interchangeable**

Geometric series-zeta function duality

Correspondence says $\frac{q^j}{1-q^j}$ and $\zeta(js)$ **are interchangeable** as gen fctns for partitions $(n, n, \ldots, n)$ (j reps / same part).

Applications

Applications

- New combinatorial bijections
- Computing coefficients of q-series, mock mod. forms
- Statistical physics
- Computational chemistry
- Computing arithmetic densities
- Computing π

Multiplicative theory of (additive) partitions

Applications

- New combinatorial bijections
- Computing coefficients of q-series, mock mod. forms
- Statistical physics
- Computational chemistry
- Computing arithmetic densities
- Computing π (quite inefficiently, to boot!)

Application: arithmetic densities

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The *arithmetic density* of a subset $S \subseteq \mathbb{Z}^+$ is

$$\lim_{N \to \infty} \frac{\#\{n \in S \mid n \le N\}}{N},$$

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Examples

- Integers $\equiv r \pmod{t}$ have density $1/t$
- Square-free integers have density $6/\pi^2 = 1/\zeta(2)$

Classical computation

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- Well-known relation between arithmetic density and zeta-type sums

Application: arithmetic densities

Classical computation

- Well-known relation between arithmetic density and zeta-type sums
- If a subset $T \subseteq \mathbb{P}$ has arith. density in $\mathbb{P}$, its density is equal to the *Dirichlet density* of T

$$\lim_{s \to 1} \frac{\sum_{p \in T} p^{-s}}{\sum_{p \in \mathbb{P}} p^{-s}}$$

Theorem (Ono-S-Wagner, 2018)

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The arith. density of integers $\equiv r \mod t$ is equal to

$$-\lim_{q \to 1} \sum_{\substack{\lambda \in \mathcal{P} \\ \mathrm{sm}(\lambda) \equiv r \pmod{t}}} \mu_{\mathcal{P}}(\lambda) q^{|\lambda|} = \frac{1}{t}.$$

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- "sm" is the smallest part of λ
- extends work of Alladi (1977), Locus Dawsey (2017)

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Application: arithmetic densities

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The arith. density of kth power-free integers is equal to

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Proofs

q-binomial thm + partition bijection + complex analysis

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- another partition-zeta connection

Application: arithmetic densities

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For d_S arith. density of a q-commensurate subset $S \subseteq \mathbb{N}$:

$$-\lim_{q\to 1} \sum_{\substack{\lambda\in\mathcal{P} \\ \mathrm{sm}(\lambda)\in S}} \mu_{\mathcal{P}}(\lambda) q^{|\lambda|} = d_S.$$

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- here $*$ is partition convolution, $\varphi(n)$ classical phi
- second formula extends work of Wang (2020)

Theorem (Ono-S-Wagner, 2020)

- Proof requires a new theory of q-series density computations based on "q-density" statistic.:

$$d_S(q) := \frac{\sum_{\mathrm{sm}(\lambda)\in S} q^{|\lambda|}}{\sum_\lambda q^{|\lambda|}} = (1-q)\sum_{\mathrm{sm}(\lambda)\in S} q^{|\lambda|}.$$

Natural number n	**Partition λ**
Prime factors of n	Parts of λ
Square-free integers	Partitions into distinct parts
$\mu(n)$	$\mu_{\mathcal{P}}(\lambda)$
$\varphi(n)$	$\varphi_{\mathcal{P}}(\lambda)$
$p_{\min}(n)$	$\mathrm{sm}(\lambda)$
$p_{\max}(n)$	$\mathrm{lg}(\lambda)$
n^{-s}	$q^{|\lambda|}$

Partition-theoretic multiverse

Philosophy of this talk (again)

- Exist multipl., division, arith. functions on partitions
- Objects in classical multiplic. number theory
  $\rightarrow$ special cases of partition-theoretic structures
- Expect arithmetic theorems $\rightarrow$ extend to partitions
- Expect partition properties $\rightarrow$ properties of integers

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Work in progress

With Akande, Beckwith, Dawsey, Hendon, Jameson, Just, Ono, Rolen, M. Schneider, Sellers, Sills, Wagner, ...

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With Akande, Beckwith, Dawsey, Hendon, Jameson, Just, Ono, Rolen, M. Schneider, Sellers, Sills, Wagner, ... you?

Gratitude

Thank you for listening :)